Heterotic Supergravity with Internal Almost-Kähler Configurations and Gauge $SO(32)$, or $E_8 \times E_8$, Instantons

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Abstract

Heterotic supergravity with (1+3)–dimensional domain wall configurations and (warped) internal, six dimensional, almost-Kähler manifolds ${}^6\mathbf{X}$ are studied. Considering on ten dimensional spacetime, nonholonomic distributions with conventional double fibrations, 2+2+...=2+2+3+3, and associated $SU(3)$ structures on internal space, we generalize for real, internal, almost symplectic gravitational structures the constructions with gravitational and gauge instantons of tanh-kink type [1, 2]. They include the first α' corrections to the heterotic supergravity action, parameterized in a form to imply nonholonomic deformations of the Yang-Mills sector and corresponding Bianchi identities. We show how it is possible to construct a variety of solutions, depending on the type of nonholonomic distributions and deformations of 'prime' instanton configurations characterized by two real supercharges. This corresponds to $\mathcal{N} = 1/2$ supersymmetric, nonholonomic manifolds from the four dimensional point of view. Our method provides a unified description of embedding nonholonomically deformed tanh-kink-type instantons into half-BPS solutions of heterotic supergravity. This allows us to elaborate new geometric methods of constructing exact solutions of motion equations, with first order α' corrections to the heterotic supergravity. Such a formalism is applied for general and/or warped almost-Kähler configurations, which allows us to generate nontrivial (1+3)-d domain walls. This formalism is utilized in our associated publication [3] in order to construct and study generic off-diagonal nonholonomic deformations of the Kerr metric, encoding contributions from heterotic supergravity.

Keywords: heterotic supergravity, almost Kähler geometry, nonholonomic (super) manifolds, nonlinear connections, domain walls.

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Contents

1 Introduction

The majority of different vacua in string gravity theories, including four dimensional spacetime domains, are elaborated with 6-d internal manifolds adapted to certain toroidal compactification or warping of extra dimensions. With the aim of obtaining interesting and realistic models of lower-dimensional physics, elaborations of 10-d theories with special Calabi–Yau (and/or more general $SU(3)$ structure) manifolds were used. Such constructions are related to pseduo-Euclidean 4-d domain configurations and warped almost-Kähler internal spaces. Recent results and reviews related to superstrings, flux compactifications, D-branes, instantons etc., are cited respectively [1, 2, 4, 5, 6, 7, 8].

Further generalizations with nontrivial solutions in the 4-d domain, such as reproductions of 4-d black hole solutions and cosmological scenarios related to modified gravity theories (MGTs) encoding information from extra dimension internal spaces, are possible if richer geometric structures are involved. Nonholonomic distributions with splitting on 4-d, 6-d and 10-d manifolds as well as almost-Kähler internal manifolds are considered when bimetric-connection structures, possible nontrivial mass terms for graviton, locally anisotropic effects etc. can be reproduced in the framework of heterotic supergravity theory.We cite [9, 10].For MGTs and their applications, we refer to [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]as well as references therein.

In a series of works [29, 30, 31, 32, 33, 34, 35, 36, 37, 38], the so-called anholonomic frame deformation method, AFDM, of constructing exact solutions in commutative and noncommutative (super) gravity and geometric flow theories has been elaborated. By straightforward analytic computations, it was proven that it is possible to decouple the gravitational field equations and generate general classes of solutions in various theories of gravity with metric, nonlinear, N-, and linear connection structures. The geometric formalism was based on spacetime fibrations determined by nonholonomic distributions with splitting of dimensions, 2 (or 3) + 2 + 2 + ... In explicit form, certain classes of N-elongated frames of reference, considered formal extensions/embeddings of 4-d spacetimes into higher dimensional spacetimes were introduced and necessary types of adapted linear connections were defined. Such connections are called distinguished, d-connections, and defined in some form that preserves the N-connection splitting. In Einstein gravity, a d-connection is considered as an auxiliary one which is additional to the Levi-Civita, LC, connection. For certain well defined conditions, the canonical d-connection can be uniquely defined by the metric structure following the conditions of metric compatibility and the conditions of zero values for "pure" horizontal and vertical components but nonzero, nonholonomically-induced, mixed vertical-horizontal torsion components. Surprisingly, such a canonical d-connection allows us to decouple the motion equations into general form. As a result, we can generate various classes of exact solutions in generalized/modified string and gravity theories. Having constructed a class of generalized solutions in explicit form (depending on generating and integration functions, generalized effective sources and integration constants), we then impose some additional constraints at the end, resulting in zero induced torsion fields. In this way, we can always "extract" solutions for LC-configurations and/or Einstein gravity. It should be emphasized that it is important to impose the zero-torsion conditions at the end, i.e. after we found a class of generalized solutions. We can not decouple and solve in general forms the corresponding systems of PDEs if we use the LC-connection from the very beginning. Here it should be noted that to work with nontrivial torsion configurations is important in order to find exact solutions in string gravity and gauge gravity models.

Using the AFDM, a series of exact and/or small parameter depending solutions were constructed, which for small deformations mimic rotoid Kerr - de Sitter like black holes/ellipsoids self–consistently embedded into generic off-diagonal backgrounds of arbitrary finite dimensions. A number of examples for 5,6 and 8 dimensional (non) commutative and/or supersymmetric spacetimes are provided, see examples in [30, 34, 35, 37, 38] and references therein. Such backgrounds can be of solitonic/ vertex / instanton type. In this paper, we develop and apply these nonholonomic geometric constructions to heterotic string theory. The motion equations are re-written in certain nonholonomic variables as generalized (effective) Einstein equations for 4-d spacetimes, encoding nontrivial geometric constructions on extra dimension internal spaces. Here we note that by using nonholonomic distributions and corresponding classes of solutions for heterotic string gravity, it is possible to mimic physically important effects in modified gravity. In a series of works [23, 39]), we studied the acceleration of the universe, certain dark energy and dark matter locally anisotropic interactions and effective renormalization of quantum gravity models via nonlinear generic off-diagonal interactions on effective Einstein spaces. In our associated publication [3] we shall elaborate in more detail the AFDM for constructing general classes of exact solutions with generic off-diagonal metrics in heterotic supergravity and generalize connections depending on all 4-d and 10-d coordinates via corresponding classes of generating and integration functions. The main results of this and the associated publications are based on the idea that we can generate physically interesting domain wall configurations (for instance, 4-d deformed black holes) by considering richer geometric structures on the internal space. We shall prove that we can parametrize and generalize all possible off-diagonal solutions in heterotic supergravity in terms of such variables with internal geometric objects that are determined by an almost-Kähler geometry. This is possible for nonholonomic distributions with conventional splitting $2+2+2=3+3$ (such constructions are based on former results in [9, 10, 40]).

In this paper, we apply methods used in the geometry of nonhlonomic and almost-Kähler manifolds in order to study heterotic supergravity derived in the low-energy limit of heterotic string theory [41, 42, 43]. We cite also section 4.4 in [1] for a summary of previous results and certain similar conventions on warped configurations and modified gravitational equations.[1] The main goal of this work is to formulate a geometric formalism which is applied in [3] for integrating in generic off-diagonal forms, and for generalized connections, the equations of

[1]Nevertheless, we shall elaborate a different system of notation with N-connections and auxiliary d–connections which allows us to define geometric objects on higher order shells of nonholonomically decomposed 10-d spacetimes.

motion of heterotic supergravity up to and including terms of order α'. The solutions of heterotic supergravity which are constructed in further sections describe (1+3)–dimensional walls endowed with generic off-diagonal metrics warped to an almost-Kähler 6-d internal space in the presence of nonholonomically deformed gravitational and gauge instantons. The generalized instanton contributions are adapted to a nontrivial, nonlinear connection structure determined by generic off-diagonal interactions which allows us to solve the Yang–Mills, YM, sector and the corresponding Bianchi identity at order α' (which is related to the gravitational constant in 10-d). Such 10-d solutions preserve two real supercharges, which correspond to the $\mathcal{N} = 1/2$ supersymmetry. The almost-Kähler internal 6-d structure can be defined for various classes of solutions in 10-d gravity if we prescribe an effective Lagrange type generating function. In such an approach, we can work both with real nonholonomic gravitational and YM instanton configurations and/or consider deformed $SU(3)$ structures.

In section 2 we provide necessary geometric preliminaries on 10-d nonholonomic manifolds with 2+2+... splitting and associated nonlinear connection (N-connection) structures related to off-diagonal metric terms and certain classes of N-adapted frames. Such a formalism is necessary for the definition of nontrivial internal space geometric structures and for the decoupling of motions equations. In section 3, the main geometric conventions for noholonomic manifolds with domain–walls, G structures and corresponding BPS equations are introduced. Section 4 is devoted to a new approach to the above with almost-Kähler geometry of internal 6-d spaces. Then in section 5, we provide solutions for nonholonomic instanton d-connections, almost-Kähler manifolds and N-adapted (effective) YM and instanton configurations. The equations of motion of heterotic supergavity are formulated in nonholonomic variables and discussed in detail. Finally, we conclude in section 6 that the heterotic supergravity theory can be formulated in nonholonomic variables which allows us to integrate in general form (see [3]) the corresponding modified Einstein equations and study stationary nonholonomic deformations of black hole solutions.

2 Nonholonomic Manifolds with 2+2+... Splitting

In this section, we summarize necessary results from the geometry of nonholonomic manifolds which will be then be applied in the heterotic supergravity (modelled in the low-energy limit of heterotic string theory as a $\mathcal{N} = 1$ and 10-d supergravity coupled to super Yang-Mills theory). A geometric formalism with nonholonomic variables and conventional 2+2+... splitting defined in such forms allows a general decoupling of motion equations will be elaborated upon, see our associated work [3]. Such a higher dimensional pseudo–Riemannian spacetime is modelled as a 10-d manifold $\mathcal{M}$, equipped with a Lorentzian metric $\breve{g}$ of signature $(++-+++++ ++)$ with a time like third coordinate.[2]. In our approach, we use a unified system of notation for straightforward applications of geometric methods for constructing exact solutions elaborated in [34, 35, 38, 36, 37]. Such notation and nonholonomic variables are different from that usually used in string theory (see, for instance, [1]). The heterotic supergravity theory is defined by a couple $(\mathcal{M}, \breve{g})$, an NS 3-form $\breve{H}$, a dilaton field $\breve{\phi}$ and a gauge connection ${}^{A}\breve{\nabla}$, with gauge group $SO(32)$ or $E_8 \times E_8$.

2.1 N-adapted frames and coordinates

For spacetime geometric models on a 10-d pseudo-Riemannian spacetime $\mathcal{M}$ with a time-like coordinate $u^3 = t$ and other coordinates being space-like, we consider conventional splitting of dimensions, $\dim \mathcal{M} = 4 + 2s = 10; s = 0, 1, 2, 3$. The AFDM allows us to construct exact solutions with arbitrary signatures of metrics $\breve{g}$, but our goal is to consider extra dimensional string gravity generalizations of the Einstein theory. In most general forms, this is possible if we use the formalism of nonlinear connection splitting for higher dimensional (super) spaces and strings which was elaborated originally in (super) Lagrange-Finsler theory [44, 45]. We shall

[2]To work with such a signature is convenient for deriving recurrent formulas for exact generic off-diagonal solutions in 4d to 10d spacetimes. By re-defining at the end of the frame/coordinate systems, we can consider "standard " coordinates and signatures of type $(-++++...+)$.

not consider Finsler type (super) gravity models in this work, but follow a similar approach with nonholonomic distributions on (super) manifolds [37, 38, 34].

Let us establish conventions on (abstract) indices and coordinates $u^{\alpha_s} = (x^{i_s}, y^{a_s})$ by labelling the oriented number of two dimensional, 2-d, "shells" added to a 4-d spacetime in GR. We consider local systems of 10-d coordinates:

$$\begin{aligned} s &= 0 : u^{\alpha_0} = (x^{i_0}, y^{a_0});\ s = 1 : u^{\alpha_1} = (x^{i_1}, y^{a_1}) = (x^{i_0}, y^{a_0}, y^{a_1}); \quad (1) \\ s &= 2 : u^{\alpha_2} = (x^{i_1} = u^{\alpha_1}, y^{a_2}) = (x^{i_0}, y^{a_0}, y^{a_1}, y^{a_2});\ s = 3 : u^{\alpha_3} = (x^{i_2} = u^{\alpha_2}, y^{a_3}) = (x^{i_0}, y^{a_0}, y^{a_1}, y^{a_2}, y^{a_3}), \end{aligned}$$

with values for indices: $i_0, j_0, ... = 1, 2; a_0, b_0, ... = 3, 4,$ when $u^3 = y^3 = t$; $a_1, b_1... = 5, 6; a_2, b_2... = 7, 8;$ $a_3, b_3... = 9, 10$; and, for instance, $i_1, j_1, ... = 1, 2, 3, 4; i_2,\ j_2, ... = 1, 2, 3, 4, 5, 6;\ i_3, j_3, ... = 1, 2, 3, 4, 5, 6, 7, 8,$ or we shall write only i_s. In brief, we shall write ${}^0u = ({}^0x,\ {}^0y)$; ${}^1u = ({}^0u,\ {}^1y) = ({}^0x,\ {}^0y,\ {}^1y)$, ${}^2u = ({}^1u,\ {}^2y) = ({}^0x,\ {}^0y,\ {}^1y,\ {}^2y)$ and ${}^2u = ({}^1u,\ {}^2y) = ({}^0x,\ {}^0y,\ {}^1y,\ {}^2y)$. In order to connect these notations to standard ones of supergravity theories (see [1, 2]),[3] We shall consider small Greek indices without subscripts, and respective coordinates x^μ, when indices $\alpha, \mu, ... = 0, 1, ..., 9$. The identification with shell coordinates is of type $x^0 = u^3 = t$ (for time-like coordinate) and (for space-like coordinates): $x^1 = u^1, x^2 = u^2, x^3 = u^4, x^4 = u^5, x^5 = u^6, x^6 = u^7, x^7 = u^8, x^8 = u^9, x^9 = u^{10}$.

Local frames/bases, e_{α_s}, on $\mathcal{M}$ are written in the form $e_{\alpha_s} = e^{\underline{\alpha}_s}_{\ \alpha_s}({}^su)\partial/\partial u^{\underline{\alpha}_s}$, where partial derivatives $\partial_{\beta_s} := \partial/\partial u^{\beta_s}$ define local coordinate bases and indices are underlined if it is necessary to emphasize that such values are defined with respect to a coordinate frame. In general, a frame e_{α_s} is nonholonomic (equivalently, anholonomic, or non-integrable), if it satisfies the anholonomy relations

$$e_{\alpha_s} e_{\beta_s} - e_{\beta_s} e_{\alpha_s} = W^{\gamma_s}_{\alpha_s \beta_s} e_{\gamma_s}.$$

In these formulas, the anholonomy coefficients $W^{\gamma_s}_{\alpha_s\beta_s} = W^{\gamma_s}_{\beta_s\alpha_s}(u)$ vanish for holonomic/integrable configurations. The dual frames, $e^{\alpha_s} = e^{\ \alpha_s}_{\underline{\alpha}_s}({}^su)du^{\underline{\alpha}_s}$, can be defined from the condition $e^{\alpha_s} \rfloor e_{\beta_s} = \delta^{\alpha_s}_{\beta_s}$. In such conditions, the 'hook' operator $\rfloor$ corresponds to the inner derivative and $\delta^{\alpha_s}_{\beta_s}$ is the Kronecker symbol.

By using nonholonomic (non–integrable) distributions, we can define 2+2+... spacetime splitting in adapted frame and coordinate forms. For our purpose, we shall work with certain distributions defining a nonlinear connection, structure via a Whitney sum

$$ {}^s\mathbf{N} : T\mathcal{M} = {}^0h\mathcal{M} \oplus {}^0v\mathcal{M} \oplus {}^1v\mathcal{M} \oplus {}^2v\mathcal{M} \oplus {}^3v\mathcal{M}. \quad (2)$$

Such a sum states a conventional horizontal (h) and vertical (v) "shell by shell" splitting. We shall write boldface letters for spaces and geometric objects enabled/adapted to a nonlinear connection structure. In local form, the nonlinear connection coefficients, $N^{a_s}_{i_s}$, are defined from a decomposition

$$ {}^s\mathbf{N} = N^{a_s}_{i_s}({}^su)dx^{i_s} \otimes \partial/\partial y^{a_s}. \quad (3)$$

A manifold $\mathcal{M}$ enabled with a nonholonomic distribution (2) is called nonholonomic (one can also use the term N–anholonomic manifold). This definition comes form the fact that in linear form, the coefficients (3)

[3]In modern gravity, the so-called ADM (Arnowit–Deser–Misner) formalism with 3+1 splitting, or any $n+1$ splitting is used, see details in [46]. It is not possible to elaborate a technique for general decoupling of the gravitational field equations and generating off-diagonal solutions in such cases because the conventional one dimensional "fibers" result in certain degenerate systems of equations. To construct exact solutions in 4 to 10 dimensional theories, it is more convenient to work with non–integrable 2+2+... splitting, see details in [37, 33, 34]. Using shell coordinates, we are able to prove in a more "compact form" certain recurrent formulas for integrating systems of nonlinear PDEs and understand important nonlinear symmetries of higher dimension spacetimes in heterotic supergravity. Such constructions are hidden in certain general statements and sophisticated formulas if we work only with "standard" indices and coordinates of type x^μ, with $\mu = 0, 1, ..., 10$.

determine a system of N–adapted local bases, with N-elongated partial derivatives, $\mathbf{e}_{\nu_s} = (\mathbf{e}_{i_s}, e_{a_s})$, and cobases with N–adapted differentials, $\mathbf{e}^{\mu_s} = (e^{i_s}, \mathbf{e}^{a_s})$, For $s=0$ (on a 4-d spacetime part)

$$\mathbf{e}_{i_0} = \frac{\partial}{\partial x^{i_0}} - N_{i_0}^{a_0}\frac{\partial}{\partial y^{a_0}},\ e_{a_0} = \frac{\partial}{\partial y^{a_0}},\ e^{i_0} = dx^{i_0}, \mathbf{e}^{a_0} = dy^{a_0} + N_{i_0}^{a_0} dx^{i_0} \text{ on } \mathbf{V} \simeq h\mathcal{M} \oplus v\mathcal{M};\ \text{or/ and} \quad (4)$$
$$\mathbf{e}_{i_s} = \frac{\partial}{\partial x^{i_s}} - N_{i_s}^{a_s}\frac{\partial}{\partial y^{a_s}},\ e_{a_s} = \frac{\partial}{\partial y^{a_s}}, e^{i_s} = dx^{i_s}, \mathbf{e}^{a_s} = dy^{a_s} + N_{i_s}^{a_s} dx^{i_s} \text{ for } s = 1,2,3.$$

The corresponding anholonomy relations with inter–shell non–integrable relations,

$$[\mathbf{e}_{\alpha_s}, \mathbf{e}_{\beta_s}] = \mathbf{e}_{\alpha_s}\mathbf{e}_{\beta_s} - \mathbf{e}_{\beta_s}\mathbf{e}_{\alpha_s} = W_{\alpha_s\beta_s}^{\gamma_s}\mathbf{e}_{\gamma_s}, \quad (5)$$

are computed $W_{i_s a_s}^{b_s} = \partial_{a_s} N_{i_s}^{b_s}$ and $W_{j_s i_s}^{a_s} = \Omega_{i_s j_s}^{a_s}$. In these formulas, the curvature of N–connection is defined as the Neijenhuis tensor, $\Omega_{i_s j_s}^{a_s} := \mathbf{e}_{j_s}\left(N_{i_s}^{a_s}\right) - \mathbf{e}_{i_s}\left(N_{j_s}^{a_s}\right)$.

2.2 d–torsions and d–curvatures of d–connections

There is a subclass of linear connections on $\mathcal{M}$, called distinguished connections, d–connections, which preserve the N–connection structure under parallelism (2).[4] With left shell labels, we write

$${}^{s}\mathbf{D} = \{\mathbf{D}_{\alpha_s}\} = ({}^{s-1}hD;\ {}^{s}vD), \text{i.e. } {}^{0}\mathbf{D} = ({}^{0}hD;\ {}^{0}vD),\ {}^{1}\mathbf{D} = ({}^{1}hD;\ {}^{1}vD),\ {}^{2}\mathbf{D} = ({}^{2}hD;\ {}^{2}vD),\ {}^{3}\mathbf{D} = ({}^{3}hD;\ {}^{3}vD).$$

In N–adapted form, the coefficients of a d–connection ${}^{s}\mathbf{D} = \{\mathbf{\Gamma}_{\beta_s\gamma_s}^{\alpha_s} = (L_{\beta_{s-1}\gamma_{s-1}}^{\alpha_{s-1}}, L_{b_s\gamma_{s-1}}^{a_s}; C_{\beta_{s-1}c_s}^{\alpha_{s-1}}, C_{b_s c_s}^{a_s})\}$, for example:

$$\begin{aligned} \mathbf{\Gamma}_{\beta_0\gamma_0}^{\alpha_0} &= (L_{j_0k_0}^{i_0}, L_{b_0k_0}^{a_0}; C_{j_0c_0}^{i_0}, C_{b_0c_0}^{a_0}), \mathbf{\Gamma}_{\beta_1\gamma_1}^{\alpha_1} = (L_{\beta_0\gamma_0}^{\alpha_0}, L_{b_1\gamma}^{a_1}; C_{\beta_0c_1}^{\alpha_0}, C_{b_1c_1}^{a_1}), \\ \mathbf{\Gamma}_{\beta_2\gamma_2}^{\alpha_2} &= (L_{\beta_1\gamma_1}^{\alpha_1}, L_{b_2\gamma_1}^{a_2}; C_{\beta_1c_2}^{\alpha_1}, C_{b_2c_2}^{a_2}), \mathbf{\Gamma}_{\beta_3\gamma_3}^{\alpha_3} = (L_{\beta_2\gamma_2}^{\alpha_2}, L_{b_3\gamma_2}^{a_3}; C_{\beta_2c_3}^{\alpha_2}, C_{b_3c_3}^{a_3}), \end{aligned} \quad (6)$$

can be computed in N–adapted form with respect to frames (4). We have to consider the equations $\mathbf{D}_{\alpha_s}\mathbf{e}_{\beta_s} = \mathbf{\Gamma}_{\beta_s\gamma_s}^{\alpha_s}\mathbf{e}_{\gamma_s}$ and covariant derivatives parametrized in the forms

$$\begin{aligned} \mathbf{D}_{\alpha_0} &= (D_{i_0}; D_{a_0}), \mathbf{D}_{\alpha_1} = (\mathbf{D}_{\alpha_0}; D_{a_1}),\ \mathbf{D}_{\alpha_2} = (\mathbf{D}_{\alpha_1}; D_{a_2}), \mathbf{D}_{\alpha_3} = (\mathbf{D}_{\alpha_2}; D_{a_3}), \\ \text{for } hD &= (L_{jk}^{i}, L_{bk}^{a}), vD = (C_{jc}^{i}, C_{bc}^{a}),\ {}^{1}hD = (L_{\beta\gamma}^{\alpha}, L_{b_1\gamma}^{a_1}),\ {}^{1}vD = (C_{\beta c_1}^{\alpha}, C_{b_1c_1}^{a_1}), \\ {}^{2}hD &= (L_{\beta_1\gamma_1}^{\alpha_1}, L_{b_2\gamma_1}^{a_2}),\ {}^{2}vD = (C_{\beta_1c_2}^{\alpha_1}, C_{b_2c_2}^{a_2}),\ {}^{3}hD = (L_{\beta_2\gamma_2}^{\alpha_2}, L_{b_3\gamma_2}^{a_3}),\ {}^{3}vD = (C_{\beta_2c_3}^{\alpha_2}, C_{b_3c_3}^{a_3}), \end{aligned}$$

or, in general form, ${}^{s}hD = (L_{\beta_{s-1}\gamma_{s-1}}^{\alpha_{s-1}}, L_{b_s\gamma_{s-1}}^{a_s}),\ {}^{s}vD = (C_{\beta_{s-1}c_s}^{\alpha_{s-1}}, C_{b_sc_s}^{a_s})$. Such coefficients can be computed with respect to mixed subsets of coordinates and/or N–adapted frames on different shells. It is always possible to consider such frame transforms when all shell frames are N-adapted, ${}^{s}D_{\alpha_{s-1}} = \mathbf{D}_{\alpha_{s-1}}$.

Using (6), we can elaborate an N–adapted covariant calculus on $\mathcal{M}$ and a corresponding differential form calculus with a differential connection 1–form $\mathbf{\Gamma}_{\beta_s}^{\alpha_s} = \mathbf{\Gamma}_{\beta_s\gamma_s}^{\alpha_s}\mathbf{e}^{\gamma_s}$ with respect to skew symmetric tensor products of N–adapted frames (4). For instance, the torsion $\mathcal{T}^{\alpha_s} = \{\mathbf{T}_{\beta_s\gamma_s}^{\alpha_s}\}$ and curvature $\mathcal{R}_{\beta_s}^{\alpha_s} = \{\mathbf{R}^{\alpha_s}{}_{\beta_s\gamma_s\delta_s}\}$ d–tensors of ${}^{s}\mathbf{D}$ can be computed in explicit form following respective formulas,

$$\mathcal{T}^{\alpha_s} := {}^{s}\mathbf{D}\mathbf{e}^{\alpha_s} = d\mathbf{e}^{\alpha_s} + \mathbf{\Gamma}_{\beta_s}^{\alpha_s} \wedge \mathbf{e}^{\beta_s} \quad (7)$$

$$\mathcal{R}_{\beta_s}^{\alpha_s} := {}^{s}\mathbf{D}\mathbf{\Gamma}_{\beta_s}^{\alpha_s} = d\mathbf{\Gamma}_{\beta_s}^{\alpha_s} - \mathbf{\Gamma}_{\beta_s}^{\gamma_s} \wedge \mathbf{\Gamma}_{\gamma_s}^{\alpha_s} = \mathbf{R}^{\alpha_s}{}_{\beta_s\gamma_s\delta_s}\mathbf{e}^{\gamma_s} \wedge \mathbf{e}^{\delta_s}, \quad (8)$$

see Refs. [37, 38, 34, 44, 45] for explicit calculi of coefficients $\mathbf{T}_{\beta_s\gamma_s}^{\alpha_s}$ and $\mathbf{R}^{\alpha_s}{}_{\beta_s\gamma_s\delta_s}$ in higher dimensions.

[4] For spaces enabled with N–connection structure, terms are used like distinguished tensor, d–tensor; distinguished spinor, d–spinor; distinguished geometric object, d–object, if the coefficients of such geometric/physical values are determined in a N–adapted shell form, with respect to frames of type (4) and their symmetric, or skew symmteric tensor products.

2.3 d–metrics and generic off–diagonal metrics

In coordinate form, a metric $\check{g}$ on $\mathcal{M}$ is written

$$ {}^{s}g = g_{\alpha_s\beta_s} e^{\alpha_s} \otimes e^{\beta_s} = g_{\underline{\alpha}_s\underline{\beta}_s} du^{\underline{\alpha}_s} \otimes du^{\underline{\beta}_s}, \tag{9}$$

for $s = 0,1,2,3$ corresponding to a conventional $2+2+...$ splitting. Under general frame transforms, the coefficients of the metric transforms as $g_{\alpha_s\beta_s} = e^{\underline{\alpha}_s}_{\ \alpha_s} e^{\underline{\beta}_s}_{\ \beta_s} g_{\underline{\alpha}_s\underline{\beta}_s}$. Similar rules can be considered for all tensor objects which do not preserve a splitting of dimensions. The same metric can be parametrized as a distinguished metric (d–metric, in boldface form), ${}^{s}\mathbf{g} = \{\mathbf{g}_{\alpha_s\beta_s}\}$,

$$\begin{aligned} {}^{s}\mathbf{g} &= g_{i_s j_s}(\ {}^{s}u)\ e^{i_s} \otimes e^{j_s} + \ g_{a_s b_s}(\ {}^{s}u)\mathbf{e}^{a_s} \otimes \mathbf{e}^{b_s} \\ &= g_{ij}(x)\ e^{i} \otimes e^{j} + g_{ab}(u)\ \mathbf{e}^{a} \otimes \mathbf{e}^{b} + g_{a_1 b_1}(\ {}^{1}u)\ \mathbf{e}^{a_1} \otimes \mathbf{e}^{b_1} + ... + \ g_{a_s b_s}(\ {}^{s}u)\mathbf{e}^{a_s} \otimes \mathbf{e}^{b_s}. \end{aligned} \tag{10}$$

Redefining (10) in coordinate frames, we find the relation between N–connection coefficients and off-diagonal metric terms in (9),

$$\begin{aligned} \underline{g}_{\alpha\beta}(\ u) &= \begin{bmatrix} g_{ij} + \ h_{ab} N_i^a N_j^b & h_{ae} N_j^e \\ h_{be} N_i^e & h_{ab} \end{bmatrix},\ \underline{g}_{\alpha_1\beta_1}(\ {}^{1}u) = \begin{bmatrix} \underline{g}_{\alpha\beta} & h_{a_1 e_1} N_{\beta_1}^{e_1} \\ h_{b_1 e_1} N_{\alpha_1}^{e_1} & h_{a_1 b_1} \end{bmatrix}, \\ \underline{g}_{\alpha_2\beta_2}(\ {}^{2}u) &= \begin{bmatrix} \underline{g}_{\alpha_1\beta_1} & h_{a_2 e_2} N_{\beta_1}^{e_2} \\ h_{b_2 e_2} N_{\alpha_1}^{e_2} & h_{a_2 b_2} \end{bmatrix},\ \underline{g}_{\alpha_3\beta_3}(\ {}^{3}u) = \begin{bmatrix} \underline{g}_{\alpha_2\beta_2} & h_{a_3 e_3} N_{\beta_2}^{e_3} \\ h_{b_3 e_3} N_{\alpha_2}^{e_3} & h_{a_3 b_3} \end{bmatrix}. \end{aligned}$$

For extra dimensions with $y^{a_s}, s \geq 1$, such coefficients $\underline{g}_{\alpha_s\beta_s}(\ {}^{s}u) = \begin{bmatrix} g_{i_s j_s} + \ h_{a_s b_s} N_{i_s}^{a_s} N_{j_s}^{b_s} & h_{a_s e_s} N_{j_s}^{e_s} \\ h_{b_s e_s} N_{i_s}^{e_s} & h_{a_s b_s} \end{bmatrix}$ are similar to those introduced in the Kaluza–Klein theory (using cylindrical compactifications on extra dimension coordinates, $N_{\alpha}^{e_s}(\ {}^{s}u) \sim A_{a_s\alpha}^{e_s}(u) y^{\alpha}$, where $A_{a_s\alpha}^{e_s}(u)$ are (non) Abelian gauge fields). In general, various parametrizations can be used for warped/trapped coordinates in extra dimension (super) gravity, string and brane gravity and modifications of GR, see examples in [47, 48, 24, 30, 29, 37, 38, 34, 35, 36, 25, 26, 27, 28].

2.4 The canonical d–connection

There are two very important linear connection structures determined by the same metric structure following geometric conditions:

$$ {}^{s}\mathbf{g} \rightarrow \left\{ \begin{array}{lll} {}^{s}\nabla: & {}^{s}\nabla\ (\ {}^{s}\mathbf{g}) = 0;\ \ {}_{\nabla}^{s}\mathcal{T} = 0, & \text{the Levi–Civita connection;} \\ {}^{s}\widehat{\mathbf{D}}: & {}^{s}\widehat{\mathbf{D}}\ (\ {}^{s}\mathbf{g}) = 0;\ h\widehat{\mathcal{T}} = 0,\ {}^{1}v\widehat{\mathcal{T}} = 0,\ {}^{2}v\widehat{\mathcal{T}} = 0,\ {}^{3}v\widehat{\mathcal{T}} = 0. & \text{the canonical d–connection.} \end{array} \right. \tag{11}$$

Let us explain how the above connections are defined:

The LC–connection ${}^{s}\nabla = \{\ {}_{\shortmid}\Gamma^{\alpha_s}_{\ \beta_s\gamma_s}\}$ can be introduced without an N–connection structure but can always be canonically distorted to a necessary type of d–connection completely defined by ${}^{s}\mathbf{g}$ following certain geometric principles. In shell N–adapted frames, we can compute ${}_{\shortmid}\mathcal{T}^{\alpha_s} = 0$ using formulas (7) for ${}^{s}\mathbf{D} \rightarrow {}^{s}\nabla$.

To elaborate on a covariant differential calculus adapted to decomposition (2) with a d–connection completely determined by the d–metric and d–connection structure, we have to consider the canonical d–connection ${}^{s}\widehat{\mathbf{D}}$ from (11). For this linear connection, the horizontal and vertical torsions are zero, i.e. $h\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}^{i}_{\ jk}\} = 0$, $v\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}^{a}_{\ bc}\} = 0$, ${}^{1}v\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}^{a_1}_{\ b_1 c_1}\} = 0, ...,$ ${}^{s}v\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}^{a_s}_{\ b_s c_s}\} = 0$. We can check using straightforward computations that such conditions are satisfied by ${}^{s}\widehat{\mathbf{D}} = \{\widehat{\mathbf{\Gamma}}^{\gamma_s}_{\ \alpha_s\beta_s}\}$ with coefficients (6) computed with respect to N–adapted frames (4) following formulas

$$\begin{aligned}
\widehat{L}^{i_s}_{j_s k_s} &= \frac{1}{2} g^{i_s r_s} \left(\mathbf{e}_{k_s} g_{j_s r_s} + \mathbf{e}_{j_s} g_{k_s r_s} - \mathbf{e}_{r_s} g_{j_s k_s} \right), \\
\widehat{L}^{a_s}_{b_s k_s} &= e_{b_s}(N^{a_s}_{k_s}) + \frac{1}{2} g^{a_s c_s} \left(\mathbf{e}_{k_s} g_{b_s c_s} - g_{d_s c_s}\ e_{b_s} N^{d_s}_{k_s} - g_{d_s b_s}\ e_{c_s} N^{d_s}_{k_s} \right), \\
\widehat{C}^{i_s}_{j_s c_s} &= \frac{1}{2} g^{i_s k_s} e_{c_s} g_{j_s k_s},\ \widehat{C}^{a_s}_{b_s c_s} = \frac{1}{2} g^{a_s d_s} \left(e_{c_s} g_{b_s d_s} + e_{c_s} g_{c_s d_s} - e_{d_s} g_{b_s c_s} \right).
\end{aligned} \tag{12}$$

The canonical d–connection ${}^s\widehat{\mathbf{D}}$ is characterized by nonholonomically induced torsion d–tensor (7) which is completely defined by ${}^s\mathbf{g}$ (10) for any chosen ${}^s\mathbf{N} = \{N^{a_s}_{i_s}\}$.[5] The N–adapted coefficients can be computed if the coefficients (12) are introduced "shell by shell" into formulas

$$\widehat{T}^{i_s}_{\ j_s k_s} = \widehat{L}^{i_s}_{j_s k_s} - \widehat{L}^{i_s}_{k_s j_s}, \widehat{T}^{i_s}_{\ j_s a_s} = \widehat{C}^{i_s}_{j_s b_s}, \widehat{T}^{a_s}_{\ j_s i_s} = -\Omega^{a_s}_{\ j_s i_s},\ \widehat{T}^{c_s}_{\ a_s j_s} = \widehat{L}^{c_s}_{a_s j_s} - e_{a_s}(N^{c_s}_{j_s}), \widehat{T}^{a_s}_{\ b_s c_s} = \ \widehat{C}^{a_s}_{b_s c_s} - \ \widehat{C}^{a_s}_{c_s b_s}. \tag{13}$$

The N-adapted formulas (12) and (13) show that any coefficient with such objects computed in 4-d can be similarly extended "shell by shell" by any value $s = 1, 2$ and 3, redefining the h- and v-indices correspondingly.

Any (pseudo) Riemannian geometry can be equivalently formulated on nonholonomic variables (${}^s\mathbf{g}$ (10), ${}^s\mathbf{N}$, ${}^s\widehat{\mathbf{D}}$) or using the standard ones (${}^s\mathbf{g}$ (9), ${}^s\nabla$). This follows from the fact that both linear connections ${}^s\nabla$ and ${}^s\widehat{\mathbf{D}}$ are defined by the same metric structure via a canonical distortion relation

$${}^s\widehat{\mathbf{D}} = \ {}^s\nabla + \ {}^s\widehat{\mathbf{Z}}. \tag{14}$$

The distorting tensor ${}^s\widehat{\mathbf{Z}} = \{ \widehat{\mathbf{Z}}^{\alpha_s}_{\ \beta_s \gamma_s} \}$ can also be constructed from the same metric ${}^s\mathbf{g}$ (10). The values $\widehat{\mathbf{Z}}^{\alpha_s}_{\ \beta_s \gamma_s}$ are algebraic combinations of $\widehat{T}^{\alpha_s}_{\ \beta_s \gamma_s}$ and vanish for zero torsion. The linear connections ${}^s\nabla$ and ${}^s\widehat{\mathbf{D}}$ are not tensor objects. It is possible to consider frame/ coordinate transforms for certain parametrization ${}^s\mathbf{N} = \{N^{a_s}_{i_s}\}$ when the conditions ${}_{\shortmid}\Gamma^{\gamma_s}_{\ \alpha_s \beta_s} = \widehat{\mathbf{\Gamma}}^{\gamma_s}_{\ \alpha_s \beta_s}$ are satisfied with respect to some N–adapted frames. In general, ${}^s\nabla \neq \ {}^s\widehat{\mathbf{D}}$ and the corresponding curvature tensors ${}_{\shortmid}R^{\alpha_s}_{\ \beta_s \gamma_s \delta_s} \neq \widehat{\mathbf{R}}^{\alpha_s}_{\ \beta_s \gamma_s \delta_s}$.

We note that we can always extract LC–configurations with zero torsion if we impose additionally for (13) the conditions

$$\widehat{\mathbf{T}}^{\gamma_s}_{\ \alpha_s \beta_s} = 0, \tag{15}$$

In general, $W^{\gamma_s}_{\alpha_s \beta_s}$ (5) may be not zero. We write this in the form ${}^s\widehat{\mathbf{D}}_{\mid \widehat{\mathcal{T}}=0} \rightarrow {}^s\nabla$. Such nonholonomic constraints may be stated in non–explicit forms and are not obligated to be described by limits of certain smooth functions.

[5] It should be noted that such a torsion is different from the torsions considered in Einstein-Cartan gauge type and string gravity theories with absolutely antisymmetric torsion. In those theories, the torsion fields are independent from the metric/vielbein one and may have proper sources. In our approach, the torsion ${}^s\widehat{\mathcal{T}}$ is completely determined by the d–metric and N–connection coefficients. There are unnecessary additional sources because such a d–torsion is determined by the nonholonomic structures. For certain additional assumptions, we can relate it's coefficients, for instance, to a subclass of nontrivial coefficients of an absolute antisymmetric torsion in string gravity.

2.5 The Riemann and Ricci d–tensors of the canonical d–connection

The N–adapted coefficients of curvature d–tensor $\mathcal{R}^{\alpha_s}_{\ \beta_s} = \{\mathbf{R}^{\alpha_s}_{\ \beta_s\gamma_s\delta_s}\}$ (8) of the canonical d–connection ${}^s\widehat{\mathbf{D}}$ (12) and ${}^s\mathbf{g}$ (10) are given by formulas

$$
\begin{aligned}
\widehat{R}^{i_s}_{\ h_s j_s k_s} &= \mathbf{e}_{k_s}\widehat{L}^{i_s}_{\ h_s j_s} - \mathbf{e}_{j_s}\widehat{L}^{i_s}_{\ h_s k_s} + \widehat{L}^{m_s}_{\ h_s j_s}\widehat{L}^{i_s}_{\ m_s k_s} - \widehat{L}^{m_s}_{\ h_s k_s}\widehat{L}^{i_s}_{\ m_s j_s} - \widehat{C}^{i_s}_{\ h_s a_s}\Omega^{a_s}_{\ k_s j_s},\\
\widehat{R}^{a_s}_{\ b_s j_s k_s} &= \mathbf{e}_{k_s}\widehat{L}^{a_s}_{\ b_s j_s} - \mathbf{e}_{j_s}\widehat{L}^{a_s}_{\ b_s k_s} + \widehat{L}^{c_s}_{\ b_s j_s}\widehat{L}^{a_s}_{\ c_s k_s} - \widehat{L}^{c_s}_{\ b_s k_s}\widehat{L}^{a_s}_{\ c_s j_s} - \widehat{C}^{a_s}_{\ b_s c_s}\Omega^{c_s}_{\ k_s j_s},\\
\widehat{P}^{i_s}_{\ j_s k_s a_s} &= e_{a_s}\widehat{L}^{i_s}_{\ j_s k_s} - \widehat{D}_{k_s}\widehat{C}^{i_s}_{\ j_s a_s} + \widehat{C}^{i_s}_{\ j_s b_s}\widehat{T}^{b_s}_{\ k_s a_s},\\
\widehat{P}^{c_s}_{\ b_s k_s a_s} &= e_{a_s}\widehat{L}^{c_s}_{\ b_s k_s} - D_{k_s}\widehat{C}^{c_s}_{\ b_s a_s} + \widehat{C}^{c_s}_{\ b_s d_s}\widehat{T}^{c_s}_{\ k_s a_s},\\
\widehat{S}^{i_s}_{\ j_s b_s c_s} &= e_{c_s}\widehat{C}^{i_s}_{\ j_s b_s} - e_{b_s}\widehat{C}^{i_s}_{\ j_s c_s} + \widehat{C}^{h_s}_{\ j_s b_s}\widehat{C}^{i_s}_{\ h_s c_s} - \widehat{C}^{h_s}_{\ j_s c_s}\widehat{C}^{i_s}_{\ h_s b_s},\\
\widehat{S}^{a_s}_{\ b_s c_s d_s} &= e_{d_s}\widehat{C}^{a_s}_{\ b_s c_s} - e_{c_s}\widehat{C}^{a_s}_{\ b_s d_s} + \widehat{C}^{e_s}_{\ b_s c_s}\widehat{C}^{a_s}_{\ e_s d_s} - \widehat{C}^{e_s}_{\ b_s d_s}\widehat{C}^{a_s}_{\ e_s c_s},
\end{aligned}
\tag{16}
$$

computed respectively for all shells $s = 0,1,2,3.$

The Ricci d–tensor $\widehat{Ric} = \{\widehat{\mathbf{R}}_{\alpha_s\beta_s} := \widehat{\mathbf{R}}^{\tau_s}_{\ \alpha_s\beta_s\tau_s}\}$ of ${}^s\widehat{\mathbf{D}}$ is introduced via a respective contraction of coefficients of the curvature tensor (16), when

$$
\widehat{\mathbf{R}}_{\alpha_s\beta_s} = \{\widehat{R}_{h_s j_s} := \widehat{R}^{i_s}_{\ h_s j_s i_s},\ \widehat{R}_{j_s a_s} := -\widehat{P}^{i_s}_{\ j_s i_s a_s},\ \widehat{R}_{b_s k_s} := \widehat{P}^{a_s}_{\ b_s k_s a_s}, \widehat{R}_{\ b_s c_s} = \widehat{S}^{a_s}_{\ b_s c_s a_s}\}. \tag{17}
$$

By contracting the N–adapted coefficients of the Ricci d–tensor with the inverse d–metric (computed as the inverse matrix of ${}^s\mathbf{g}$ (10)), we define and compute the scalar curvature of ${}^s\widehat{\mathbf{D}}$,

$$
\begin{aligned}
{}^s\widehat{R} &:= \mathbf{g}^{\alpha_s\beta_s}\widehat{\mathbf{R}}_{\alpha_s\beta_s} = g^{i_s j_s}\widehat{R}_{i_s j_s} + g^{a_s b_s}\widehat{R}_{a_s b_s}\\
&= \widehat{R} + \widehat{S} + {}^1\widehat{S} + {}^2\widehat{S} + {}^3\widehat{S},
\end{aligned}
\tag{18}
$$

with respective h– and v–components of scalar curvature, $\widehat{R} = g^{ij}\widehat{R}_{ij},\ S = g^{ab}S_{ab},\ {}^1S = g^{a_1b_1}S_{a_1b_1},\ {}^2S = g^{a_2b_2}S_{a_2b_2},\ {}^3S = g^{a_3b_3}R_{a_3b_3}.$

3 Nonholonomic Domain–Walls in Heterotic Supergravity

In order to find more general classes of solutions and a more realistic phenomenology in string theory, with more complex vacua, we shall work with internal nonholonomic manifolds with a richer geometric structure (with nontrivial N–connections, effective extra dimensional gravitational and matter field interactions etc.). Such nonhlonomic manifolds preserve less supersymmetry and lead to more realistic models. This follows from the holonomy principle which can be formulated for the parallel d–spinor equations (we omit such considerations for ${}^s\widehat{\mathbf{D}}$, which can be found in [44, 49, 47, 48]).

The goal of this subsection is to formulate a model of the $\mathcal{N} = 1$ and 10-d supergravity coupled to super Yang–Mills theory working with a 10–d nonholonomic manifold with $2+2+...+2 = 10$ splitting by a N–connection structure and with a gauge d–connection ${}^s_A\widehat{\mathbf{D}}$ distorting in nonholonomic form similarly to (14) the connection ${}^A\check{\nabla}$ with gauge group $SO(32)$ or $E_8\times E_8$. Our model of nonholonomic heterotic supergravity theory will be determined by $(\mathcal{M}, \mathbf{N}, {}^s\mathbf{g}, \widehat{\mathbf{H}}, \widehat{\phi}, {}^s_A\widehat{\mathbf{D}})$, where $\mathbf{N}$ and ${}^s\mathbf{g}$ are respective, N–connection and d–metric; $\widehat{\mathbf{H}}$ is the nonholonomic version of NS 3-form $\check{H}$; $\widehat{\phi}$ is a dilaiton field on an N–anholonomic manifold which transform into the standard one $\check{\phi}$ for LC–configurations; ${}^s_A\widehat{\mathbf{D}}$ is an N–adapted version of ${}^A\check{\nabla}$ (we shall describe this construction below).

3.1 The canonical d–connection and BPS equations

By letting ${}^s\widehat{\mathbf{D}}_{|\widehat{\mathcal{T}}=0} \to {}^s\nabla$, see (15), and formulating in N–adapted form the anomaly cancellation condition of the 10-d super Yang–Mills, YM, theory coupled to $\mathcal{N} = 1$, 10-d supergravity can be written as a Bianchi identity on $\widehat{\mathbf{H}}$,

$$\widehat{\mathbf{d}}\widehat{\mathbf{H}} = \frac{\alpha'}{4} Tr(\widehat{\mathbf{F}} \wedge \widehat{\mathbf{F}} - \widetilde{\mathbf{R}} \wedge \widetilde{\mathbf{R}}), \quad (19)$$

implying two curvature 2-forms: $\widehat{\mathbf{F}}$ is the strength of the gauge d–connection and ${}^{s}_{A}\widehat{\mathbf{D}}; \widetilde{\mathbf{R}}$ is the strength of a d–connection $\widetilde{\mathbf{D}}$ which will be defined below; $\widehat{\mathbf{d}}$ is the 10-d exterior derivative; Tr means the trace on gauge group indices. There are different connections which can be used in anomaly cancellation conditions of type (19). This depends on the type of renormalization scheme and the preferences of string theory physicists, see discussions in [50, 51, 52, 53] and references therein.[6] In non N–adapted form, $\widetilde{\mathbf{D}}$ was considered. For instance as $\widetilde{\mathbf{D}} \to \widetilde{\nabla} = {}^{+}\widetilde{\nabla}$, where ${}^{\pm}\widetilde{\nabla} = \nabla \pm \check{H}$, with absolute antisymmetric torsion $\check{H}$. In supergravity, the connection $\widetilde{\nabla}$ is determined by imposing the instanton equation $\widetilde{R} \cdot \epsilon = 0$. In this paper, we adopt the point of view that $\widetilde{\mathbf{D}} = {}^{s}\widehat{\mathbf{D}} + \widehat{\mathbf{H}}$ when $\widetilde{\mathbf{D}}$ can be transformed into an almost-Kähler d–connection studied in details in Refs. [10, 9, 54, 40], which allows us to find a number of nontrivial generic off–diagonal solutions in 10-d following the AFDM. The instanton equation can be written in nonholonomic form, $\widetilde{\mathbf{R}} \cdot \epsilon = 0$.

In string theory, it is considered that for $\widehat{\mathbf{H}} = 0$ the internal manifold should be Ricci flat and Kähler and such a condition does not stablize all Kähler moduli. A non-zero 3-form flux of the above breaks scale invariance and provides a corresponding stabilization [55]. In a similar form, the canonical d–connection structure may result in stable configurations for corresponding nonholonomic constraints. For instance, the α'–solutions with nonzero $\check{H}$ constructed in [1] preserve a $\mathcal{N} = 1/2$ supersymmetry (with two real supercharges instead of four for the usual $\mathcal{N} = 1$ supersymmetry) in $1 + 3$ external dimensions. This is implied by the so called Bogomol'nyi-Prasad-Sommerfield, BPS, conditions halving the supersymmetry by the presence of a domain wall, see details in references [50, 51, 52, 53, 1].

At the zeroth order in α', zero value of the NS 3-form $\check{H}$ and a dilaton field $\check{\phi} = const$, the PBS configurations are given by

$$\check{\mathcal{M}} = \mathbb{R}^{2,1} \times \check{c}(\ {}^{6}X); \check{H} = 0, \check{\phi} = const,$$

where $\check{c}(\ {}^{6}X)$ is the metric cone over a 6-d almost-Kähler manifold ${}^{6}X$. In this formula, $\mathbb{R}^{2,1}$ is a pseudo-Euclidean space with signature $(++-)$ and local coordinates $x^{\check{i}} = (x^{i}, y^{3} = t)$, for $\check{i}, \check{j}, ... = 1, 2, 3$; the local coordinates on ${}^{6}X$ are those used for the shells $s = 1, 2, 3$, when

$$u^{\check{a}} = y^{\check{a}} = \{u^{a_1} = y^{a_1} = (y^5, y^6), u^{a_2} = y^{a_2} = (y^7, y^8), u^{a_3} = y^{a_3} = (y^9, y^{10})\},$$

with indices $\check{a}, \check{b}, ... = 5, 6, 7, 8, 9, 10$, are labeled in a form compatible with coordinate conventions (1). The space-like coordinate $u^4 = y^4$ will be used for a 7-d warped extension of ${}^{6}X \to {}^{7}X = (y^4, {}^{6}X)$ with local coordinates on ${}^{7}X$,

$$y^{\widetilde{a}} = (y^4, y^{a_1}, y^{a_2}, y^{a_3}) = (y^4, y^5, y^6, y^7, y^8, y^9, y^{10})$$

for indices $\widetilde{a}, \widetilde{b}, ... = 4, 5, ...10$.

3.2 Conventions for 10-d nonholonomic manifolds, d–tensor and d–spinor indices

In heterotic superstring theory, one considers domain walls for nonholonomic splitting transform, in general, in nonholonomic domains. In the framework of conventions for indices and coordinates on shells (1), we consider

[6]It important to note here that the constant α' is related to the gravitational constant ${}_{10}\kappa$ in 10-d following formulas $({}_{10}\kappa)^2 = \frac{1}{4\pi}(4\pi^2\alpha')^4 = \frac{\ell_{str}^8}{4\pi}$. The 4-d gravitational constant ${}_{4}\kappa$ is given by $({}_{4}\kappa)^2 = \ell_{str}^{-6}\ ({}_{10}\kappa)^2 = M_{Pl}^{-2} = 8\pi G^{-1}$, where G is the Newton constant.

additional parametrizations

$$\begin{array}{rcl} \mu_s, \alpha_s, ... &=& 1,2,...10, \text{ on a 10-d nonholonomic manifold with shell coordinates } u^{\mu_s} = (x^i, y^a, y^{a_1}, y^{a_2}, y^{a_3}) \text{ on } \mathcal{M}; \\ \check{\mu}, \check{\alpha}, ... &=& 1,2,...10, \text{ equivalently, with general indices and coordinates } u^{\check{\mu}} \text{ on } \mathcal{M}; \\ \check{i}, \check{j}, ... &=& 1,2,3, \text{ for a 3-d pseudo-Euclidean signature } (++-) \text{ and coordinates } u^{\check{i}} = x^{\check{i}} = (x^i, y^3 = t) = (x^1, x^2, t); \\ \check{a}, \check{b}, ... &=& 5,6,7,8,9,10, \text{ with } u^{\check{a}} = y^{\check{a}} \text{ on N-anholonomic } {}^6\mathbf{X} \text{ with Euclidian signature ;} \\ \widetilde{a}, \widetilde{b}, ... &=& 4,5,6,7,8,9,10, \text{ with } u^{\widetilde{a}} = y^{\widetilde{a}} \text{ on N-anholonomic } {}^7\mathbf{X} \text{ with Euclidian signature ;} \end{array}$$

We shall underline such indices in order to emphasize that we work in a coordinate base $\partial_{\underline{\check{a}}}$: For instance, $A^{\check{a}}\mathbf{e}_{\check{a}} = A^{\underline{\check{a}}}\partial_{\underline{\check{a}}}$ if we consider decompositions of a d–vector.

Anti–symmetrization is performed with a factorial factor, for instance, $A_{[\check{\mu}}B_{\check{\alpha}]} := \frac{1}{2}(A_{\check{\mu}}B_{\check{\alpha}} - B_{\check{\alpha}}A_{\check{\mu}})$. A p–form ω is expressed $\omega = \frac{1}{p!}\omega_{\check{\mu}_1...\check{\mu}_p}\mathbf{e}^{\check{\mu}_1} \wedge ... \wedge \mathbf{e}^{\check{\mu}_p}$, see (4). The Clifford action of such a form on a spinor ϵ can be defined in N–adapted form as

$$\omega \cdot \epsilon := \frac{1}{p!}\omega_{\check{\mu}_1...\check{\mu}_p}\gamma^{\check{\mu}_1...\check{\mu}_p}\epsilon,$$

for $\gamma^{\check{\mu}_1...\check{\mu}_p} := \gamma^{[\check{\mu}_1}...\gamma^{\check{\mu}_p]}$ if the Clifford d–algebra is introduced following convention

$$\{\gamma^{\check{\mu}}, \gamma^{\check{\nu}}\} := \gamma^{\check{\mu}}\gamma^{\check{\nu}} + \gamma^{\check{\nu}}\gamma^{\check{\mu}} = 2\mathbf{g}^{\check{\mu}\check{\nu}}$$

for 10-d gamma matrices $\gamma^{\check{\mu}}$ with $\mathbf{g}^{\check{\mu}\check{\nu}} = \{\mathbf{g}^{\mu_s\nu_s}\}$ admitting a shell decomposition (10). A theory of N–adapted spinors and Dirac operators is elaborated upon [44, 47, 48, 49], see also references therein. We omit such considerations in this work.

It is proven in [2] that at first order in α', the BPS equations are solved by ${}^A\check{\nabla} = {}^c\nabla, \check{H} = 0, \check{\phi} = const$, where ${}^c\nabla$ is the LC–connection on $\check{c}({}^6X)$. A series of less trivial solutions with $\check{H} \neq 0$ have been studied in [2, 56, 57, 58, 59, 1] under assumptions that the gauge field is chosen to be an instanton and within the framework of dynamic $SU(3)$ structures. In this work, we shall extend those results in string theory by proving that the equations of motion of heterotic string supergravity can be decoupled and solved in very general off–diagonal forms with dependence, in principle, of all 10-d spacetime cooridinates (using the AFDM for higher dimensions [37, 38, 34]). In order to preserve certain relations to former "holonomic" solutons, we shall work with N–anholonomic manifolds $\mathcal{M}$ and ${}^6\mathbf{X}$ determined by d–metric (10). We shall prove that for any solution of certain generalized/modified 10-d Einstein equations for ${}^s\widehat{\mathbf{D}}_{|\widehat{\mathcal{T}}=0} \to {}^s\nabla$ and additional extensions with gauge and scalar fields, the submanifold ${}^6\mathbf{X}$ can be endowed with almost-Kähler variables [9, 54, 40]. In result, the curvature $\widetilde{\mathbf{R}}$ of d–connection $\widetilde{\mathbf{D}}$ can be determined for arbitrary solutions of the equations of motion of heterotic supergravity with generic off–diagonal interactions, nonholonomically deformed connections and various parametrizations of effective sources.

3.3 Nonholonomic domain–wall backgrounds

In Einstein–Cartan and gauge field theories, torsion fields have certain sources subjected to algebraic equations. If a background with vanishing fermionic vacuum expectations does not have prescribed nonholonomic distributions, the supersymmetry transformations of the corresponding fermionic fields are zero. Such conditions for the holonomic backgrounds are known as BPS equations. Up to and including terms of order α' for N–anholonomic backgrouns, the BPS equations are formulated for a Majorana–Weyl spinor ϵ,

$$\begin{array}{rcl} (\widehat{\mathbf{D}} - \widehat{\mathbf{H}}) \cdot \epsilon &=& 0, \\ (\widehat{\mathbf{d}}\widehat{\phi} - \frac{1}{2}\widehat{\mathbf{H}}) \cdot \epsilon &=& 0, \\ \widehat{\mathbf{F}} \cdot \epsilon &=& 0. \end{array} \quad (20)$$

We note that in this work hatted boldface objects denote 10–d geometric/physical objects on N–anholonomic manifolds enabled with d–connection structure $\widehat{\mathbf{D}}$.

In heterotic string gravity, (see details and references in [1]) backgrounds given by 4-d domain walls with 6 internal directions stating a compact manifolds ${}^{6}X$ with $SU(3)$ structure for a 10-d metric ansatz with quadratic element

$$\begin{aligned} ds^2[\mathring{g}] &= \mathring{g}_{\check{\alpha}\check{\beta}}(y^4, y^{\check{a}})du^{\check{\alpha}}du^{\check{\beta}} = \mathring{g}_{\alpha_s\beta_s}(y^4, y^{\check{a}})du^{\alpha_s}du^{\beta_s} = \\ &= e^{2A(y^4,y^{\check{a}})}\left[\left(dx^1\right)^2 + \left(dx^2\right)^2 - \left(dy^3\right)^2 + e^{2B(y^{\check{a}})}\left(dy^4\right)^2 + \mathring{g}_{\check{b}\check{c}}(y^4, y^{\check{a}})dy^{\check{b}}dy^{\check{c}}\right], \end{aligned} \tag{21}$$

are considered, where $y^3 = t$ and y^4 is chosen to be transverse to the domain wall enabled with coordinates (x^i, t). We can consider orthonormal frames $e^{\check{a}} = e^{\check{a}}(y^4, y^{\check{c}})e^{\check{a}'}$ for a prescribed N–connection structure $\mathring{\mathbf{N}} = \{\mathring{N}_{i_s}^{a_s}(y^4, y^{c_s}) \to \mathring{N}_{i_s}^{\check{a}}(y^4, y^{\check{c}})\}$ defined for a local system of coordinates in the internal 6-d manifold ${}^{6}X$ embedded via warped coordinate y^4 into higher dimensional ones and transform it into N-anholonomic manifold ${}^{6}\mathbf{X} \subset {}^{7}\mathbf{X} \subset \mathcal{M}$, endowed with a d–metric structure of type (10),

$$\begin{aligned} ds^2[\mathring{\mathbf{g}}] &= e^{2\mathring{A}(y^4,y^{\check{a}})}[\left(dx^1\right)^2 + \left(dx^2\right)^2 - \left(\mathring{\mathbf{e}}^3\right)^2 + e^{2\mathring{B}(y^{\check{a}})}\left(\mathring{\mathbf{e}}^4\right)^2 + \\ & \mathring{g}_{a_1}(y^4, y^{a_1})\left(\mathring{\mathbf{e}}^{a_1}\right)^2 + \mathring{g}_{a_2}(y^4, y^{a_2})\left(\mathring{\mathbf{e}}^{a_2}\right)^2 + \mathring{g}_{a_3}(y^4, y^{a_3})\left(\mathring{\mathbf{e}}^{a_3}\right)^2], \end{aligned} \tag{22}$$

where

$$\begin{aligned} a_0 &= 3: \ \mathring{\mathbf{e}}^3 = dt + \mathring{n}_{i_0}dx^{i_0}, \text{ for } \mathring{N}_{i_0}^3 = \mathring{n}_{i_0}(x^k, y^4), \text{ for } k, i_0 = 1,2; \\ a_0 &= 4: \ \mathring{\mathbf{e}}^4 = dy^4 + \mathring{w}_{i_0}dx^{i_0}, \text{ for } \mathring{N}_{i_0}^4 = \mathring{w}_{i_0}(x^k, y^4); \\ a_1 &= 5: \ \mathring{\mathbf{e}}^5 = dy^5 + \mathring{n}_{i_1}dx^{i_1}, \text{ for } \mathring{N}_{i_1}^5 = \mathring{n}_{i_1}(x^k, y^4, y^6), \text{ for } i_1 = 1,2,3,4; \\ a_1 &= 6: \ \mathring{\mathbf{e}}^6 = dy^6 + \mathring{w}_{i_1}dx^{i_1}, \text{ for } \mathring{N}_{i_1}^6 = \mathring{w}_{i_1}(x^k, y^4, y^6); \\ a_2 &= 7: \ \mathring{\mathbf{e}}^7 = dy^7 + \mathring{n}_{i_2}dx^{i_2}, \text{ for } \mathring{N}_{i_2}^7 = \mathring{n}_{i_2}(x^k, y^4, y^6, y^8), \text{ for } i_2 = 1,2,3,4,5,6; \\ a_2 &= 8: \ \mathring{\mathbf{e}}^8 = dy^8 + \mathring{w}_{i_2}dx^{i_2}, \text{ for } \mathring{N}_{i_2}^8 = \mathring{w}_{i_2}(x^k, y^4, y^5, y^6, y^8); \\ a_3 &= 9: \ \mathring{\mathbf{e}}^9 = dy^9 + \mathring{n}_{i_3}dx^{i_3}, \text{ for } \mathring{N}_{i_3}^9 = \mathring{n}_{i_3}(x^k, y^4, y^5, y^6, y^7, y^8, y^{10}), \text{ for } i_3 = 1,2,3,4,5,6,7,8; \\ a_3 &= 10: \ \mathring{\mathbf{e}}^{10} = dy^{10} + \mathring{w}_{i_3}dx^{i_3}, \text{ for } \mathring{N}_{i_3}^{10} = \mathring{w}_{i_3}(x^k, y^4, y^5, y^6, y^7, y^8, y^{10}). \end{aligned} \tag{23}$$

In this paper, we shall study stationary configurations with Killing symmetry on ∂_t when the metric/d–metric ansatz do not depend on coordinate $y^3 = t$. We call an ansatz $\mathring{g}$ (21) as a prime off–diagonal metric and an ansatz $\mathring{\mathbf{g}}$ (22) (prime metrics will be labeled by a circle $\circ$).[7]

The overall goal of this and associated [3] articles is to study nonholonomic deformations of a primary metric $\mathring{\mathbf{g}}$ into a target metric ${}^{s}\mathbf{g}$,

$$\begin{aligned} \mathring{\mathbf{g}} &= [\mathring{\mathbf{g}}_{\alpha_s}, \mathring{N}_{i_s}^{a_s}] \to {}^{s}\mathbf{g} = [\mathbf{g}_{\alpha_s} = \eta_{\alpha_s}\mathring{\mathbf{g}}_{\alpha_s}, N_{i_s}^{a_s} = \eta_{i_s}^{a_s}\mathring{N}_{i_s}^{a_s}], \\ &\to {}_{\varepsilon}^{s}\mathbf{g} = [\mathbf{g}_{\alpha_s} = (1+\varepsilon\chi_{\alpha_s})\mathring{\mathbf{g}}_{\alpha_s}, N_{i_s}^{a_s} = (1+\varepsilon\chi_{i_s}^{a_s})\mathring{N}_{i_s}^{a_s}], \\ \text{for } \eta_{\alpha_s} &\simeq 1+\varepsilon\chi_{\alpha_s}, \eta_{i_s}^{a_s}, \eta_{i_s}^{a_s} \simeq 1+\varepsilon\chi_{i_s}^{a_s}, \text{ where } 0 \leq \varepsilon \ll 1. \end{aligned} \tag{24}$$

In these formulas, we do not consider summations on repeating indices. Any target metric ${}^{s}\mathbf{g}$ will be subjected to these conditions to define new classes of solutions for certain systems of nonlinear PDEs in heterotic string

[7] It should be emphasized that indices $a_1 = (5,6), a_2 = (7,8), a_3 = (9,10)$ are shell adapted but indices $\check{a}, \check{c}, \check{a}'$ may take, in general, values $6,7,...10$ with shell mixing of indices. In order to apply the AFDM, we shall always consider certain frame/coordinate transforms with N–adapted shell redefinitions of interior indices and coordinates. Indices with "inverse" hats are convenient for parametrization of almost-Kähler structures but shell indices are important for constructing exact off-diagonal solutions in 10-d gravity.

gravity (see next section) or in a MGT. It is always possible to model the internal 6-d space as an almost-Kähler manifold. For certain subclasses of solutions with $\lim_{\varepsilon\to 0}\ {}^{s}\mathbf{g}\to \mathring{\mathbf{g}}$, the η–polarization functions $(\eta_{\alpha_s}, \eta^{a_s}_{i_s})\to 1$. In general, such limits with a small parameter ε may not exist, or can behave singularly.

In our next constructions, we show how G structures [58, 59] can be adapted to N–connections. The Killing spinor ϵ is adapted to above prime and target d-metric ansatz as

$$\epsilon(x^i,t,y^4,y^{\check{a}})=\rho(x^i,t)\otimes\eta(y^4,y^{\check{a}})\otimes\check{\theta},$$

where the domain wall spinor ρ has two real components corresponding to the two real supercharges which the holonomic background $\mathbb{R}^{2,1}$ preserves; η is a covariantly constant Majorana spinor on ${}^{7}\mathbf{X}=(y^4,\ {}^{6}\mathbf{X})$; and $\check{\theta}$ is an eigenvector of the respective Pauli matrix.

It is possible to preserve, in the tangent bundle $T\mathcal{M}$, the $(1+2)$–dimensional Lorentz invariance together with N–connection splitting if restrictions on $\widehat{\phi}$ and $\widehat{\mathbf{H}}_{\alpha_s\beta_s\mu_s}$:

$$\mathbf{e}_{\check{i}}\widehat{\phi}=0,\widehat{\mathbf{H}}_{\ \check{i}\widetilde{a}\widetilde{b}}=0,\widehat{\mathbf{H}}_{\ \check{i}\check{j}\widetilde{b}}=0. \tag{25}$$

are considered. For a flat 3-d Minkowski spacetime, the only non-zero components of the NS 3-form flux are $\widehat{\mathbf{H}}_{4\check{a}\check{b}},\widehat{\mathbf{H}}_{\check{a}\check{b}\check{c}}$ and

$$\widehat{\mathbf{H}}_{\check{i}\check{j}\check{k}}=\ell\sqrt{|\mathbf{g}_{\check{i}\check{j}}|}\epsilon_{\check{i}\check{j}\check{k}}$$

for $\ell=const$ and the totally antisymmetric tensor on $\mathbb{R}^{2,1}$ with normalization $\epsilon_{123}=1$.

Let us define G_2 d–structure adapted to the N–connection splitting in ${}^{7}\mathbf{X}=\mathbb{R}\times\ {}^{6}\mathbf{X}$ enabled with an arbitrary d–metric (of type included in (21) and (22)), respectively,

$$\begin{aligned} ds^2[\ {}^{7}\mathring{g}] &= e^{2B(y^{\check{a}})}\left(dy^4\right)^2+\mathring{g}_{\check{b}\check{c}}(y^4,y^{\check{a}})dy^{\check{b}}dy^{\check{c}} \text{ and} \\ ds^2[{}^{7}\mathring{\mathbf{g}}] &= e^{2\mathring{B}(y^{\check{a}})}\left(\mathring{\mathbf{e}}^4\right)^2+\mathring{g}_{a_1}(y^4,y^{a_1})\left(\mathring{\mathbf{e}}^{a_1}\right)^2+\mathring{g}_{a_2}(y^4,y^{a_2})\left(\mathring{\mathbf{e}}^{a_2}\right)^2+\mathring{g}_{a_3}(y^4,y^{a_2},y^{a_3})\left(\mathring{\mathbf{e}}^{a_3}\right)^2. \end{aligned} \tag{26}$$

Such holonomic structures were studied in [58, 59] for certain special parametrizations of functions $B,\mathring{B}$ and $\mathring{g}_{a_s}$. For nonholonomic deformations (24), we generated d–metrics on ${}^{7}\mathbf{X}$ with $\mathring{g}_4=e^{2\mathring{B}(y^{\check{a}})}\to g_4=e^{2B(y^{\check{a}})}$ (via coordinate transforms, we can consider parametrizations with $B=0$ but we keep a nontrivial value of B in order to compare our results with those for holonomic Kähler configurations outlined in [1], where Δ is considered for B),

$$ds^2[{}^{7}\mathbf{g}]=[e^{2B(y^{\check{a}})}\left(\mathbf{e}^4\right)^2+g_{a_1}(y^4,y^{a_1})\left(\mathbf{e}^{a_1}\right)^2+g_{a_2}(y^4,y^{a_2})\left(\mathbf{e}^{a_2}\right)^2+g_{a_3}(y^4,y^{\check{a}})\left(\mathbf{e}^{a_3}\right)^2].$$

In this formula and (26), the internal spaces, the indices and coordinates can be written in any form we need for the definition of almost-Kähler or diadic shell structures as we explained above in footnote 7. The d–metric ${}^{7}\mathbf{g}$ defines the Hodge operator $*_7$. The G_2 d–structure is given by a 3-form $\varpi\in\wedge^3(\ {}^{7}\mathbf{X})$ and its 7-d Hodge dual $\mathcal{W}:=\ {}^{7}*\varpi\in\wedge^4(\ {}^{7}\mathbf{X})$ (in a similar form, d-structures can be introduced for d–spinors and 7-d gamma matrices). In N–adapted differential form with absolute differential operator ${}^{7}\mathbf{d}$ on ${}^{7}\mathbf{X}$, the BPS equations (20) imply

$$\begin{aligned} {}^{7}\mathbf{d}\varpi &= 2\ {}^{7}\mathbf{d}\widehat{\phi}\wedge\varpi-\ {}^{7}*\widehat{\mathbf{H}}-\ell\mathcal{W},\ {}^{7}\mathbf{d}\mathcal{W}=2\ {}^{7}\mathbf{d}\widehat{\phi}\wedge\mathcal{W}, \\ {}^{7}*\ {}^{7}\mathbf{d}\widehat{\phi} &= -\frac{1}{2}\widehat{\mathbf{H}}\wedge\varpi,\ {}^{7}*\ell=2\widehat{\mathbf{H}}\wedge\mathcal{W}. \end{aligned} \tag{27}$$

Such relations can be written in N-adapted form with respect to frames (4) and for the canonical d–connection ${}^{s}\widehat{\mathbf{D}}$.

4 Almost-Kähler Internal Configurations in Heterotic Supergravity

We analyze important geometric structures which can be defined for the decomposition ${}^7\mathbf{X} = \mathbb{R}\times {}^6\mathbf{X}$. For holonomic distributions, it is always possible to rewrite the equation (27) in terms of an $SU(3)$ structure defined on ${}^6\mathbf{X}$ and the domain wall direction. Such constructions are related to the complex structure and Kähler geometry [58, 59, 1]. In order to elaborate a heterotic theory with generic off–diagonal metrics $\mathbf{g} = ({}^4\mathbf{g}, {}^6\mathbf{g})$ (10), we need a richer, real geometric structure for internal space ${}^6\mathbf{X}$. Up to certain classes of frame transforms, any ${}^6\mathbf{g}$ of Euclidean signature can be uniquely related to an almost-Kähler geometry. The approach was considered for elaborating certain methods of deformation of the Einstein and Finsler modified gravity theories and for formulating models of almost-Kähler geometric flows and Lie algebroid structures, see [9, 10, 54, 40]. The goal of this subsection is to apply those methods in the study of heterotic string gravity models.

4.1 Almost symplectic structures induced by effective Lagrange distributions

In order to enable the internal space with a complex Kähler structure, one considers a decomposition of the Majorana spinor η into two 6-d spinors of definite chirality, $\eta = \frac{1}{\sqrt{2}}(\eta_+ + \eta_-)$. We can specifiy any $SU(3)$ structure on ${}^6\mathbf{X}$ via a couple of geometric objects (J,θ) a real 2-form J and a complex 3-form $\theta = \theta_+ + i\theta_-$, where $i^2 = -1$. Such values can be defined for any fixed values y^4 and by using the chiral spinors $\eta_\pm$. The relation between holonomic G_2 structure $(\varpi, \mathcal{W})$ and (J,θ) is studied in [1, 58, 59, 60].

The real nonholonomic almost-Kähler geometry is also determined by a couple $(\tilde{J},\tilde{\theta})$ which in our work is constructed to be uniquely determined by a N–connection structure $\tilde{N}$ and a 6-d metric ${}^6\mathbf{g} \rightarrow \tilde{\theta}$ defined by a Lagrange type distribution $\tilde{L}(y^4, y^{\check{a}})$. In this case, we also have a nontrivial nonolonomically induced torsion with conventional splitting of internal coordinates and indices in the form $y^{\check{a}} = (y^{\acute{i}}, y^{\grave{a}})$, where $\acute{i}, \acute{j}, ... = 5,6,7$ and (for conventional "vertical", v, indices) $\grave{a}, \grave{b}, ... = 8,9,10$.

We can re-parametrize a general Riemannian metric ${}^6\mathbf{g} \subset \mathbf{g}$ on ${}^6\mathbf{X}$, when possible dependencies on 4-d spacetime coordinates ${}^0u = (x^i, y^a)$ are considered as parameters [for simplicity, we shall omit writting of 4-d coordinates if it will not result in ambiguities],

$$
\begin{aligned}
ds^2[\,{}^6g] &= g_{\check{b}\check{c}}(x^i, y^a, y^{\check{a}})dy^{\check{b}}dy^{\check{c}} \text{ and/or} \\
ds^2[{}^6\mathring{\mathbf{g}}] &= g_{a_1}(\,{}^0u, y^{a_1})\,(\mathbf{e}^{a_1})^2 + g_{a_2}(\,{}^1u, y^{a_1})\,(\mathbf{e}^{a_2})^2 + g_{a_3}(\,{}^2u, y^{a_2})\,(\mathbf{e}^{a_3})^2 .
\end{aligned}
$$

in the form:

$$
\begin{aligned}
{}^6\mathbf{g} &= g_{\acute{i}\acute{j}}(\,{}^0u, y^{\acute{i}}, y^{\grave{a}})dy^{\acute{i}} \otimes dy^{\acute{j}} + g_{\grave{a}\grave{b}}(\,{}^0u, y^{\acute{i}}, y^{\grave{a}})e^{\grave{a}} \otimes e^{\grave{b}}, \\
\mathbf{e}^{\grave{a}} &= dy^{\grave{a}} - N_{\acute{i}}^{\grave{a}}(u)dy^{\acute{i}}.
\end{aligned}
\tag{28}
$$

In formluas (28), the vierbein coefficients $e^{\check{a}}{}_{\underline{\check{a}}}$ of the dual basis $e^{\check{a}} = (e^{\acute{i}}, e^{\grave{a}}) = e^{\check{a}}{}_{\underline{\check{a}}}(u)dy^{\underline{\check{a}}}$, are parametrized to define a formal $3+3$ splitting with N–connection structure ${}^6\mathbf{N} = \{N_{\acute{i}}^{\grave{a}}\}$.

It is possible to prescribe any generating function $L(u) = L(x^i, y^a, y^{\check{a}})$ on ${}^6\mathbf{X}$ with nondegenerate Hessian $\det|\,\tilde{g}_{\grave{a}\grave{b}}| \neq 0$ for

$$
\tilde{g}_{\grave{a}\grave{b}} := \frac{1}{2}\frac{\partial^2 L}{\partial y^{\grave{a}}\partial y^{\grave{b}}}. \tag{29}
$$

We define a canonical N–connection structure

$$
\begin{aligned}
\tilde{N}_{\acute{i}}^{\grave{a}} &= \frac{\partial G^{\grave{a}}}{\partial y^{7+\acute{i}}}, \\
G^{\grave{a}} &= \frac{1}{4}\,\tilde{g}^{\grave{a}\ 7+\acute{i}}\left(\frac{\partial^2 L}{\partial y^{7+\acute{i}}\partial y^{\acute{k}}}y^{7+\acute{k}} - \frac{\partial L}{\partial y^{\acute{i}}}\right).
\end{aligned}
\tag{30}
$$

In these formulas, $\tilde{g}^{\grave{a}\grave{b}}$ is inverse to $\tilde{g}_{\grave{a}\grave{b}}$ and respective contractions of $h-$ and v–indices, $\acute{\imath},\acute{j},...=5,6,7$ and $\grave{a},\grave{b},...=8,9,10$, are performed following this rule: For example, we take an up v–index $\grave{a}=3+\acute{\imath}$ and contract it with a low index $\acute{\imath}=1,2,3$. Using (29) and (30), we construct an internal 6-d d–metric

$$\begin{aligned} {}^{6}\widetilde{\mathbf{g}} &= \widetilde{g}_{\acute{\imath}\acute{j}}dy^{\acute{\imath}}\otimes dy^{\acute{j}}+\ \widetilde{g}_{\grave{a}\grave{b}}\widetilde{\mathbf{e}}^{\grave{a}}\otimes\widetilde{\mathbf{e}}^{\grave{b}}, \\ \widetilde{\mathbf{e}}^{\grave{a}} &= dy^{\grave{a}}+\tilde{N}_{\acute{\imath}}^{\grave{a}}dy^{\acute{\imath}},\ \{\ \widetilde{g}_{\grave{a}\grave{b}}\}=\ \{\widetilde{g}_{7+\acute{\imath}\ 7+\acute{j}}\}. \end{aligned} \quad (31)$$

It should be emphasized that any d-metric ${}^{6}\mathbf{g}$ (28) can be parametrized by coefficients ${}^{6}\mathbf{g}_{\check{a}\check{b}}=[g_{\acute{\imath}\acute{j}},g_{\grave{a}\grave{b}},N_{\acute{\imath}}^{\grave{a}}]$ computed with respect to a N-adapted basis $\mathbf{e}^{\check{a}}=(e^{\acute{\imath}}=dy^{\acute{\imath}},\mathbf{e}^{\grave{a}})$ which is related to the metric ${}^{6}\widetilde{\mathbf{g}}_{\check{a}\check{b}}=[\widetilde{g}_{\acute{\imath}\acute{j}},\widetilde{g}_{\grave{a}\grave{b}},\tilde{N}_{\acute{\imath}}^{\grave{a}}]$ (31) with coefficients defined with respect to a N–adapted dual basis $\widetilde{\mathbf{e}}^{\check{a}}=(e^{\acute{\imath}}=dy^{\acute{\imath}},\widetilde{\mathbf{e}}^{\grave{a}})$ if the conditions ${}^{6}\mathbf{g}_{\check{a}'\check{b}'}e^{\check{a}'}_{\ \check{a}}e^{\check{b}'}_{\ \check{b}}=\ {}^{6}\widetilde{\mathbf{g}}_{\check{a}\check{b}}$ related to corresponding frame transfoms are satisfied. Fixing any values ${}^{6}\mathbf{g}_{\check{a}'\check{b}'}$ and ${}^{6}\widetilde{\mathbf{g}}_{\check{a}\check{b}}$, we have to solve a system of quadratic algebraic equations with unknown variables $e^{\check{a}'}_{\ \check{a}}$. A nonholonomic $2+2+2=3+3$ splitting of ${}^{6}\mathbf{X}$ with ${}^{6}\mathbf{g}_{\check{a}\check{b}}=[g_{\acute{\imath}\acute{j}},g_{\grave{a}\grave{b}},N_{\acute{\imath}}^{\grave{a}}]$ is convenient for constructing generic off-diagonal solutions but similar N–connection splitting with equivalent ${}^{6}\widetilde{\mathbf{g}}_{\check{a}\check{b}}=[\widetilde{g}_{\acute{\imath}\acute{j}},\widetilde{g}_{\grave{a}\grave{b}},\tilde{N}_{\acute{\imath}}^{\grave{a}}]$ will allow us to define real solutions for effective EYMH systems.

A set of coefficients $\tilde{\mathbf{N}}=\{\tilde{N}_{\acute{\imath}}^{\grave{a}}\}$ defines an N–connection splitting as a Whitney sum,

$$T\ {}^{6}\mathbf{X}=h\ {}^{6}\mathbf{X}\oplus v\ {}^{6}\mathbf{X} \quad (32)$$

into conventional internal horizontal (h) and vertical (v) subspaces. In local form, this can be written as

$$\tilde{\mathbf{N}}=\tilde{N}_{\acute{\imath}}^{\grave{a}}(u)dy^{\acute{\imath}}\otimes\frac{\partial}{\partial y^{\grave{a}}}, \quad (33)$$

with ${}^{6}\mathbf{X}={}^{3+3}\mathbf{X}$. As a result, there are N–adapted frame (vielbein) structures,

$$\widetilde{\mathbf{e}}_{\check{a}}=\left(\widetilde{\mathbf{e}}_{\acute{\imath}}=\frac{\partial}{\partial y^{\acute{\imath}}}-\tilde{N}_{\acute{\imath}}^{\grave{a}}\frac{\partial}{\partial y^{\grave{a}}},e_{\grave{a}}=\frac{\partial}{\partial y^{\grave{a}}}\right), \quad (34)$$

with dual frame (coframe) structures $\widetilde{\mathbf{e}}^{\check{a}}$, see (31). These vielbein structures define the nonholonomy relations

$$[\widetilde{\mathbf{e}}_{\check{a}},\widetilde{\mathbf{e}}_{\check{b}}]=\widetilde{\mathbf{e}}_{\check{a}}\widetilde{\mathbf{e}}_{\check{b}}-\widetilde{\mathbf{e}}_{\check{b}}\widetilde{\mathbf{e}}_{\check{a}}=\widetilde{w}_{\check{a}\check{b}}^{\check{c}}\widetilde{\mathbf{e}}_{\check{c}} \quad (35)$$

with (antisymmetric) anholonomy coefficients $\widetilde{w}_{\acute{\imath}\grave{a}}^{\grave{b}}=\partial_{\grave{a}}\tilde{N}_{\acute{\imath}}^{\grave{b}}$ and $\widetilde{w}_{\acute{j}\acute{\imath}}^{\grave{a}}=\widetilde{\Omega}_{\acute{\imath}\acute{j}}^{\grave{a}}$, where $\widetilde{\Omega}_{\acute{\imath}\acute{j}}^{\grave{a}}=\widetilde{\mathbf{e}}_{\acute{j}}\left(\tilde{N}_{\acute{\imath}}^{\grave{a}}\right)-\widetilde{\mathbf{e}}_{\acute{\imath}}\left(\tilde{N}_{\acute{j}}^{\grave{a}}\right)$ are the coefficients of N–connection curvature (defined as the Neijenhuis tensor).

Using the canonical N–connection splitting, we introduce a linear operator $\widetilde{\mathbf{J}}$ acting on vectors on ${}^{6}\mathbf{X}$ following formulas $\widetilde{\mathbf{J}}(\mathbf{e}_{\acute{\imath}})=-e_{7+\acute{\imath}}$ and $\widetilde{\mathbf{J}}(e_{7+\acute{\imath}})=\widetilde{\mathbf{e}}_{\acute{\imath}}$, where $\widetilde{\mathbf{J}}\circ\widetilde{\mathbf{J}}=-\mathbb{I}$, for $\mathbb{I}$ being the unity matrix, and construct a tensor field,

$$\begin{aligned} \widetilde{\mathbf{J}} &= \widetilde{\mathbf{J}}^{\check{a}}_{\ \check{b}}\ e_{\check{a}}\otimes e^{\check{b}}=\widetilde{\mathbf{J}}^{\underline{\check{a}}}_{\ \underline{\check{b}}}\ \frac{\partial}{\partial u^{\underline{\check{a}}}}\otimes du^{\underline{\check{b}}} \\ &= \widetilde{\mathbf{J}}^{\check{a}'}_{\ \beta'}\ \widetilde{\mathbf{e}}_{\check{a}'}\otimes\widetilde{\mathbf{e}}^{\beta'}=-e_{7+\acute{\imath}}\otimes e^{\acute{\imath}}+\widetilde{\mathbf{e}}_{\acute{\imath}}\otimes\ \widetilde{\mathbf{e}}^{7+\acute{\imath}} \\ &= -\frac{\partial}{\partial y^{\acute{\imath}}}\otimes dx^{\acute{\imath}}+\left(\frac{\partial}{\partial y^{\acute{\imath}}}-\ \widetilde{N}_{\acute{\imath}}^{\grave{a}}\frac{\partial}{\partial y^{\grave{a}}}\right)\otimes\left(dy^{7+\acute{\imath}}+\widetilde{N}_{\acute{k}}^{7+\acute{\imath}}dy^{\acute{k}}\right). \end{aligned} \quad (36)$$

The corresponding d-tensor field defined globally by an almost complex structure on ${}^{6}\mathbf{X}$ is completely determined by a prescribed generating function $\tilde{L}(y^{4},y^{\check{a}})\subset L(u^{\underline{\alpha}_s})$. In this subsection, we consider only structures $\mathbf{J}=\tilde{\mathbf{J}}$ induced by a $\tilde{N}_{\acute{k}}^{7+\acute{\imath}}$. In general, we can define an almost complex structure $\mathbf{J}$ for an arbitrary N–connection

$\mathbf{N}$, stating a nonholonomic $3+3$ splitting by using N–adapted bases (34) which can be included (if necessary) into respective nonholonomic frames of the 10-d spacetime, see (4).

The Neijenhuis tensor field for $\tilde{\mathbf{J}}$ (equivalently, the curvature of N–connection $\tilde{\mathbf{N}}$) is

$$^{\mathbf{J}}\widetilde{\mathbf{\Omega}}(\mathbf{X},\mathbf{Y}) := -[\mathbf{X},\mathbf{Y}] + [\tilde{\mathbf{J}}\mathbf{X},\tilde{\mathbf{J}}\mathbf{Y}] - \tilde{\mathbf{J}}[\tilde{\mathbf{J}}\mathbf{X},\mathbf{Y}] - \tilde{\mathbf{J}}[\mathbf{X},\tilde{\mathbf{J}}\mathbf{Y}], \quad (37)$$

for any d–vectors $\mathbf{X},\mathbf{Y} \in T\ {}^{6}\mathbf{X}$. With respect to N–adapted bases (34), a subset of the coefficients of the Neijenhuis tensor defines the N–connection curvature,

$$\widetilde{\Omega}_{\acute{i}\acute{k}}^{\grave{a}} = \frac{\partial \widetilde{N}_{\acute{i}}^{\grave{a}}}{\partial y^{\acute{k}}} - \frac{\partial \widetilde{N}_{\acute{k}}^{\grave{a}}}{\partial y^{\acute{i}}} + \widetilde{N}_{\acute{i}}^{\grave{b}} \frac{\partial \widetilde{N}_{\acute{k}}^{\grave{a}}}{\partial y^{\grave{b}}} - \widetilde{N}_{\acute{k}}^{\grave{b}} \frac{\partial \widetilde{N}_{\acute{i}}^{\grave{a}}}{\partial y^{\grave{b}}}. \quad (38)$$

The nonholonomic structure is integrable if $\widetilde{\Omega}_{\acute{i}\acute{k}}^{\grave{a}} = 0$. We get a complex structure if and only if both the h– and v–distributions are integrable, i.e. if and only if

$$\widetilde{\Omega}_{\acute{i}\acute{k}}^{\grave{a}} = 0 \text{ and } \frac{\partial \widetilde{N}_{\acute{i}}^{\grave{a}}}{\partial y^{\acute{k}}} - \frac{\partial \widetilde{N}_{\acute{k}}^{\grave{a}}}{\partial y^{\acute{i}}} = 0.$$

An almost symplectic structure on a manifold is introduced by a nondegenerate 2–form

$$\theta = \frac{1}{2}\theta_{\check{a}\check{b}}(u) e^{\check{a}} \wedge e^{\check{b}} = \frac{1}{2}\theta_{\acute{i}\acute{k}}(u) e^{\acute{i}} \wedge e^{\acute{k}} + \frac{1}{2}\theta_{\grave{a}\grave{b}}(u) \mathbf{e}^{\grave{a}} \wedge \mathbf{e}^{\grave{b}}.$$

An almost Hermitian model of an internal 6-d Riemannian space equipped with a N–connection structure $\mathbf{N}$ is defined by a triple $\mathbf{H}^{3+3} = ({}^{6}\mathbf{X}, \theta, \mathbf{J})$, where $\theta(\mathbf{X},\mathbf{Y}) \doteqdot \mathbf{g}(\mathbf{J}\mathbf{X},\mathbf{Y})$ for any $\mathbf{g}$ (28). A space $\mathbf{H}^{3+3}$ is almost-Kähler, denoted $\mathbf{K}^{3+3}$, if and only if $d\theta = 0$.

Using $\mathbf{g} = \tilde{\mathbf{g}}$ (31) and structures $\tilde{\mathbf{N}}$ (30) and $\tilde{\mathbf{J}}$, we define

$$\widetilde{\theta}(\mathbf{X},\mathbf{Y}) := \widetilde{\mathbf{g}}\left(\tilde{\mathbf{J}}\mathbf{X},\mathbf{Y}\right),$$

for any d–vectors $\mathbf{X},\mathbf{Y} \in T\ {}^{6}\mathbf{X}$. In local N–adapted form, we have

$$\begin{aligned} \widetilde{\theta} &= \frac{1}{2}\ \widetilde{\theta}_{\check{a}\check{b}}(u) e^{\check{a}} \wedge e^{\check{b}} = \frac{1}{2}\ \widetilde{\theta}_{\underline{\check{a}}\underline{\check{b}}}(u) du^{\underline{\check{a}}} \wedge du^{\underline{\check{b}}} \\ &= \tilde{g}_{\acute{i}\acute{k}}(x^{i}, y^{a}, y^{\check{a}}) e^{7+\acute{i}} \wedge dy^{\acute{k}} = \tilde{g}_{\acute{i}\acute{j}}(x^{i}, y^{a}, y^{\check{a}})(dy^{7+\acute{i}} + \tilde{N}_{\acute{k}}^{7+\acute{i}} dy^{\acute{k}}) \wedge dy^{\acute{j}}. \end{aligned} \quad (39)$$

Considering the form

$$\widetilde{\omega} = \frac{1}{2}\frac{\partial \widetilde{L}}{\partial y^{7+\acute{i}}} dy^{\acute{i}},$$

we prove by a straightforward computation that $\widetilde{\theta} = d\ \widetilde{\omega}$, i.e. $d\ \widetilde{\theta} = dd\ \widetilde{\omega} = 0$. As a result, any canonical effective Lagrange structure $\left(\mathbf{g} = \tilde{\mathbf{g}},\ \tilde{\mathbf{N}},\tilde{\mathbf{J}}\right)$ induces an almost-Kähler geometry. The 2–form (39) can be written

$$\begin{aligned} \theta &= \widetilde{\theta} = \frac{1}{2}\widetilde{\theta}_{\acute{i}\acute{j}}(u) e^{\acute{i}} \wedge e^{\acute{j}} + \frac{1}{2}\ \widetilde{\theta}_{\grave{a}\grave{b}}(u) \widetilde{\mathbf{e}}^{\grave{a}} \wedge \widetilde{\mathbf{e}}^{\grave{b}} \\ &= g_{\acute{i}\acute{j}}(u) \left[dy^{\acute{i}} + \widetilde{N}_{\acute{k}}^{7+\acute{i}}(u) dy^{\acute{k}}\right] \wedge dy^{\acute{j}}, \end{aligned} \quad (40)$$

where the nontrivial coefficients $\widetilde{\theta}_{\grave{a}\grave{b}} = \widetilde{\theta}_{7+\acute{i}\ 7+\acute{j}}$ are equal to the N-adapted coefficients $\widetilde{\theta}_{\acute{i}\ \acute{j}}$ respectively.

4.2 Almost symplectic connections for N-anhlonomic internal spaces

Taking a general 2–form θ constructed for any metric $\mathbf{g}$ and almost complex $\mathbf{J}$ structures on ${}^6\mathbf{X}$, one obtains $d\theta \neq 0$. Nevertheless, we can always define a $3+3$ splitting induced by an effective Lagrange generating function when $d\ \widetilde{\theta} = 0$. Considering frame transforms, $\theta_{\check{a}'\check{b}'}e^{\check{a}'}_{\ \check{a}}e^{\check{b}'}_{\ \check{b}} = \widetilde{\theta}_{\check{a}\check{b}}$, we can write $d\ \theta = 0$ for any set of 2–form coefficients related via frame transforms to a canonical symplectic structure.

There is a unique normal d–connection

$$\begin{aligned} \widetilde{\mathbf{D}} &= \left\{ h\widetilde{D} = (\widetilde{D}_{\acute{k}}, {}^v\widetilde{D}_{\acute{k}} = \widetilde{D}_{\acute{k}}); v\widetilde{D} = (\widetilde{D}_{\grave{c}}, {}^v\widetilde{D}_{\grave{c}} = \widetilde{D}_{\grave{c}}) \right\} \\ &= \{ \widetilde{\mathbf{\Gamma}}^{\check{a}}_{\check{b}\check{c}} = (\widetilde{L}^{\acute{\imath}}_{\acute{j}\acute{k}}, {}^v\widetilde{L}^{7+\acute{\imath}}_{7+\acute{j}\ 7+\acute{k}} = \widetilde{L}^{\acute{\imath}}_{\acute{j}\acute{k}};\ \widetilde{C}^{\acute{\imath}}_{\acute{j}\grave{c}} = {}^v\widetilde{C}^{7+\acute{\imath}}_{7+\acute{j}\ \grave{c}}, {}^v\widetilde{C}^{\grave{a}}_{\grave{b}\grave{c}} = \widetilde{C}^{\grave{a}}_{\grave{b}\grave{c}}) \}, \end{aligned} \tag{41}$$

which is metric compatible, $\widetilde{D}_{\acute{k}}\widetilde{g}_{\acute{\imath}\acute{j}} = 0$ and $\widetilde{D}_{\grave{b}}\ \widetilde{g}_{\acute{\imath}\acute{j}} = 0$, and completely defined by a couple of h– and v–components $\widetilde{\mathbf{D}}_{\check{a}} = (\widetilde{D}_{\acute{k}}, \widetilde{D}_{\grave{b}})$. The corresponding N–adapted coefficients $\widetilde{\mathbf{\Gamma}}^{\check{a}}_{\check{b}\gamma} = (\widetilde{L}^{\acute{\imath}}_{\acute{j}\acute{k}}, {}^v\widetilde{C}^{\grave{a}}_{\grave{b}\grave{c}})$ are given by

$$\widetilde{L}^{\acute{\imath}}_{\acute{j}\acute{k}} = \frac{1}{2}\ \widetilde{g}^{\acute{\imath}\acute{h}}\left(\widetilde{\mathbf{e}}_{\acute{k}}\widetilde{g}_{\acute{j}\acute{h}} + \widetilde{\mathbf{e}}_{\acute{j}}\widetilde{g}_{\acute{h}\acute{k}} - \widetilde{\mathbf{e}}_{\acute{h}}\widetilde{g}_{\acute{j}\acute{k}}\right), \widetilde{C}^{\acute{\imath}}_{\acute{j}\acute{k}} = \frac{1}{2}\ \widetilde{g}^{\acute{\imath}\acute{h}}\left(\frac{\partial \widetilde{g}_{\acute{j}\acute{h}}}{\partial y^{\acute{k}}} + \frac{\partial \widetilde{g}_{\acute{h}\acute{k}}}{\partial y^{\acute{j}}} - \frac{\partial \widetilde{g}_{\acute{j}\acute{k}}}{\partial y^{\acute{h}}}\right). \tag{42}$$

To elaborate a differential form calculus on ${}^6\mathbf{X}$ which is adapted to the canonical N–connection $\widetilde{\mathbf{N}}$, we introduce the normal d–connection 1–form

$$\widetilde{\mathbf{\Gamma}}^{\acute{\imath}}_{\acute{j}} = \widetilde{L}^{\acute{\imath}}_{\acute{j}\acute{k}}e^{\acute{k}} + \widetilde{C}^{\acute{\imath}}_{\acute{j}\acute{k}}\mathbf{e}^{\acute{k}}. \tag{43}$$

Using this linear connection, we prove that the Cartan structure equations are satisfied,

$$de^{\acute{k}} - e^{\acute{j}} \wedge \widetilde{\mathbf{\Gamma}}^{\acute{\imath}}_{\acute{j}} = -\widetilde{\mathcal{T}}^{\acute{k}},\ d\mathbf{e}^{\acute{k}} - \mathbf{e}^{\acute{j}} \wedge \widetilde{\mathbf{\Gamma}}^{\acute{\imath}}_{\acute{j}} = -\ {}^v\widetilde{\mathcal{T}}^{\acute{\imath}}, \tag{44}$$

and

$$d\widetilde{\mathbf{\Gamma}}^{\acute{\imath}}_{\acute{j}} - \widetilde{\mathbf{\Gamma}}^{\acute{h}}_{\acute{j}} \wedge \widetilde{\mathbf{\Gamma}}^{\acute{\imath}}_{\acute{h}} = -\widetilde{\mathcal{R}}^{\acute{\imath}}_{\ \acute{j}}. \tag{45}$$

The h– and v–components of the torsion 2–form $\widetilde{\mathcal{T}}^{\check{a}} = \left(\widetilde{\mathcal{T}}^{\acute{k}}, {}^v\widetilde{\mathcal{T}}^{\acute{\imath}}\right) = \widetilde{\mathbf{T}}^{\check{a}}_{\ \check{c}\check{b}}\ \widetilde{\mathbf{e}}^{\check{c}} \wedge \widetilde{\mathbf{e}}^{\check{b}}$ from (44) are computed

$$\widetilde{\mathcal{T}}^{\acute{\imath}} = \widetilde{C}^{\acute{\imath}}_{\acute{j}\acute{k}}e^{\acute{j}} \wedge \mathbf{e}^{\acute{k}},\ {}^v\widetilde{\mathcal{T}}^{\acute{\imath}} = \frac{1}{2}\ \widetilde{\Omega}^{\acute{\imath}}_{\acute{k}\acute{j}}e^{\acute{k}} \wedge e^{\acute{j}} + (\frac{\partial\ \widetilde{N}^{\acute{\imath}}_{\acute{k}}}{\partial y^{\acute{j}}} - \widetilde{L}^{\acute{\imath}}_{\acute{j}\acute{k}})e^{\acute{k}} \wedge \mathbf{e}^{\acute{j}}. \tag{46}$$

In these formulas, $\widetilde{\Omega}^{\acute{\imath}}_{\acute{k}\acute{j}}$ are coefficients of the curvature of $\widetilde{N}^{\acute{\imath}}_{\acute{k}}$ defined by formulas similar to (38). The formulas (46) parametrize the h– and v–components of torsion $\widetilde{\mathbf{T}}^{\check{a}}_{\ \check{c}\check{b}}$ in the form

$$\widetilde{T}^{\acute{\imath}}_{\acute{j}\acute{k}} = 0, \widetilde{T}^{\acute{\imath}}_{\acute{j}\grave{a}} = \widetilde{C}^{\acute{\imath}}_{\acute{j}\grave{a}}, \widetilde{T}^{\grave{a}}_{\acute{k}\acute{j}} = \ \widetilde{\Omega}^{\acute{\imath}}_{\acute{k}\acute{j}}, \widetilde{T}^{\grave{a}}_{\acute{\imath}\grave{b}} = e_{\grave{b}}\left(\widetilde{N}^{\grave{a}}_{\acute{\imath}}\right) - \widetilde{L}^{\grave{a}}_{\ \grave{b}\acute{\imath}}, \widetilde{T}^{\grave{a}}_{\grave{b}\grave{c}} = 0. \tag{47}$$

We emphasize that $\widetilde{\mathbf{T}}$ vanishes on h- and v–subspaces, i.e. $\widetilde{T}^{\acute{\imath}}_{\acute{j}\acute{k}} = 0$ and $\widetilde{T}^{\grave{a}}_{\grave{b}\grave{c}} = 0$, but other nontrivial h–v–components are induced by the nonholonomic structure determined canonically by $\mathbf{g} = \widetilde{\mathbf{g}}$ (31) and $\tilde{L}$.

An explicit calculus of the curvature 2–form from (45) results in

$$\widetilde{\mathcal{R}}^{\acute{\imath}}_{\ \acute{j}} = \widetilde{\mathbf{R}}^{\acute{\imath}}_{\ \acute{j}\check{c}\check{b}}\ \mathbf{e}^{\check{c}} \wedge\ \mathbf{e}^{\check{b}} = \frac{1}{2}\widetilde{R}^{\acute{\imath}}_{\ \acute{j}\acute{k}\acute{h}}e^{\acute{k}} \wedge e^{\acute{h}} + \widetilde{P}^{\acute{\imath}}_{\ \acute{j}\acute{k}\grave{a}}e^{\acute{k}} \wedge \widetilde{\mathbf{e}}^{\grave{a}} + \frac{1}{2}\ \widetilde{S}^{\acute{\imath}}_{\ \acute{j}\grave{c}\grave{d}}\widetilde{\mathbf{e}}^{\grave{c}} \wedge \widetilde{\mathbf{e}}^{\grave{d}}. \tag{48}$$

The corresponding nontrivial N–adapted coefficients of curvature $\widetilde{\mathbf{R}}^{\check{a}}_{\ \ \check{c}\check{b}\check{e}}$ of $\widetilde{\mathbf{D}}$ are

$$\begin{aligned} \widetilde{R}^{\acute{\imath}}_{\ \acute{j}\acute{k}\acute{h}} &= \widetilde{\mathbf{e}}_{\acute{k}}\widetilde{L}^{\acute{\imath}}_{\ \acute{h}\acute{j}} - \widetilde{\mathbf{e}}_{\acute{j}}\widetilde{L}^{\acute{\imath}}_{\ \acute{h}\acute{k}} + \widetilde{L}^{\acute{m}}_{\ \acute{h}\acute{j}}\widetilde{L}^{\acute{\imath}}_{\ \acute{m}\acute{k}} - \widetilde{L}^{\acute{m}}_{\ \acute{h}\acute{k}}\widetilde{L}^{\acute{\imath}}_{\ \acute{m}\acute{j}} - \widetilde{C}^{\acute{\imath}}_{\ \acute{h}\grave{a}}\ \widetilde{\Omega}^{\grave{a}}_{\ \acute{k}\acute{j}} \\ \widetilde{P}^{\acute{\imath}}_{\ \acute{j}\acute{k}\grave{a}} &= e_{\grave{a}}\widetilde{L}^{\acute{\imath}}_{\ \acute{j}\acute{k}} - \widetilde{\mathbf{D}}_{\acute{k}}\widetilde{C}^{\acute{\imath}}_{\ \acute{j}\grave{a}}, \widetilde{S}^{\grave{a}}_{\ \grave{b}\grave{c}\grave{d}} = \ e_{\grave{d}}\widetilde{C}^{\grave{a}}_{\ \grave{b}\grave{c}} - e_{\grave{c}}\widetilde{C}^{\grave{a}}_{\ \grave{b}\grave{d}} + \widetilde{C}^{\grave{e}}_{\ \grave{b}\grave{c}}\widetilde{C}^{\grave{a}}_{\ \grave{e}\grave{d}} - \widetilde{C}^{\grave{e}}_{\ \grave{b}\grave{d}}\widetilde{C}^{\grave{a}}_{\ \grave{e}\grave{c}}. \end{aligned}$$

By definition, the Ricci d–tensor $\widetilde{\mathbf{R}}_{\breve{b}\breve{c}} = \widetilde{\mathbf{R}}^{\breve{a}}{}_{\breve{b}\breve{c}\breve{a}}$ is computed

$$\widetilde{\mathbf{R}}_{\breve{b}\breve{c}} = \left(\widetilde{R}_{\acute{i}\acute{j}}, \widetilde{R}_{\acute{i}\grave{a}}, \widetilde{R}_{\grave{a}\acute{i}}, \widetilde{R}_{\grave{a}\grave{b}} \right). \quad (49)$$

The scalar curvature $\widetilde{R}$ of $\widetilde{\mathbf{D}}$ is given by two h- and v–terms,

$$\widetilde{R} = \widetilde{\mathbf{g}}^{\breve{b}\breve{c}} \widetilde{\mathbf{R}}_{\breve{b}\breve{c}} = \widetilde{g}^{\acute{i}\acute{j}} \widetilde{R}_{\acute{i}\acute{j}} + \widetilde{g}^{\grave{a}\grave{b}} \widetilde{R}_{\grave{a}\grave{b}}. \quad (50)$$

The normal d–connection $\widetilde{\mathbf{D}}$ (41) defines a canonical almost symplectic d–connection, $\widetilde{\mathbf{D}} \equiv {}_{\theta}\widetilde{\mathbf{D}}$, which is N–adapted to the effective Lagrange and, related to almost symplectic structures, i.e. it preserves the splitting under parallelism (32),

$$\widetilde{\mathbf{D}}_{\mathbf{X}}\ \widetilde{\mathbf{g}} = {}_{\theta}\widetilde{\mathbf{D}}_{\mathbf{X}}\ \widetilde{\theta} = 0,$$

for any $\mathbf{X} \in T\ {}^{6}\mathbf{X}$ and its torsion is constrained to satisfy the conditions $\widetilde{T}^{\acute{i}}_{\acute{j}\acute{k}} = 0$ and $\widetilde{T}^{\grave{a}}_{\grave{b}\grave{c}} = 0$.

We conclude that having chosen a regular generating function $L(x,y)$ on a Riemannian internal space $\mathbf{V}$, we can always model this spacetime equivalently as an-almost Kähler manifold. Using corresponding nonholonomic frame transforms and deformation of connections, we can work with equivalent geometric data on the internal space ${}^{6}\mathbf{X}$, for convenience

$$({}^{6}\mathbf{g}, {}^{6}\mathbf{N}, {}^{6}\widehat{\mathbf{D}}) \iff (\tilde{\mathbf{g}}, \tilde{\mathbf{N}}, \widetilde{\mathbf{D}}, \widetilde{L}) \iff (\widetilde{\theta}, \widetilde{\mathbf{J}}, {}_{\theta}\widetilde{\mathbf{D}}).$$

The first N-adapted model is convenient for constructing exact solutions in 6-d and 10-d gravity models (this will be addressed in the associated paper [3], see also examples in [37]). The second nonholonomic model with "tilde" geometric objects (with so–called Lagrange-Finsler variables, in our case, on a 6–d Riemannian space) is an example of an internal space with nontrivial nonholonomic $3+3$ splitting by a canonical N–connection structure determined by an effective Lagrange function $\widetilde{L}$. The $(\widetilde{\theta}, \widetilde{\mathbf{J}}, {}_{\theta}\widetilde{\mathbf{D}})$ defines an almost-Kähler geometric model on ${}^{6}\mathbf{X}$ with nontrivial nonholonomically induced d–torsion structure $\widetilde{\mathcal{T}}^{\breve{a}}$. This way, we can mimic a complex like differential geometry by real values and elaborate on various applications to quantum gravity, string/brane and geometric flow theories [9, 10, 40]. Introducing the complex imaginary unit $i^2 = -1$, with $\widetilde{\mathbf{J}} \approx i$... and integrable nonholonomic distributions, we can redefine the geometric constructions for complex manifolds. Using nonholonomic real $3+3$ distributions, we can elaborate gravitational and gauge like models of internal spaces, for instance, with $SO(3)$, or $SU(3)$ symmetries and their tensor products. Two different approaches can be unified in a geometric language with double nonholonomic fibrations $2+2+2 = 3+3$. Any d–metric with internal $2+2+2$ nonholonomic splitting can be redefined by nonholonomic frame transforms into an almost symplectic structure with $3+3$ decomposition. Considering actions of $SO(3)$, or $SU(3)$ on corresponding tangent spaces, we can reproduce all results with Kähler internal spaces related 4-d, 6-d and 10-d solutions for holonomic configurations obtained in Refs. [1, 2].

4.3 N-adapted $\mathbf{G}_2$ structures on almost-Kähler internal spaces

For any 3–form $\Theta = \Theta_{\breve{a}\breve{b}\breve{c}} \widetilde{\mathbf{e}}^{\breve{a}} \wedge \widetilde{\mathbf{e}}^{\breve{b}} \wedge \widetilde{\mathbf{e}}^{\breve{c}}$ on ${}^{6}\mathbf{X}$ endowed with a canonical almost complex structure $\tilde{\mathbf{J}}$ (36)

$$\Theta = {}^{+}\Theta + \tilde{\mathbf{J}}\ {}^{-}\Theta. \quad (51)$$

We can fix these conditions such that for $\tilde{\mathbf{J}} \to i, i^2 = -1, \Theta$ defines an $SU(3)$ structure defined on ${}^{6}\mathbf{X}$ and the tangent space to the domain wall with $\Theta = {}^{+}\Theta + i\ {}^{-}\Theta$. Defining the gamma matrices $\gamma_{\breve{a}}$ on ${}^{6}\mathbf{X}$ from the relation $\widetilde{\gamma}_{\breve{a}}\widetilde{\gamma}_{\breve{b}} + \widetilde{\gamma}_{\breve{b}}\widetilde{\gamma}_{\breve{a}} = 2\ {}^{6}\widetilde{\mathbf{g}}_{\breve{a}\breve{b}}$, see (31), we can relate the geometric objects in the almost-Kähler model of the internal space to the models with $SU(3)$ structure on a typical fiber in the tangent bundle $T\ {}^{6}\mathbf{X}$. For integrable $SU(3)$ structures and Kähler internal spaces, one works with the structure forms $(J, \underline{\Theta})$, when

$$\underline{\Theta}_{\breve{a}\breve{b}\breve{c}} = (\eta_{+})^{\dagger} \underline{\gamma}_{\breve{a}\breve{b}\breve{c}} \eta_{-} \text{ and } J_{\breve{a}\breve{b}} = \mp(\eta_{\pm})^{\dagger} \underline{\gamma}_{\breve{a}\breve{b}} \eta_{\pm}$$

are considered for a Kähler metric $2\ {}^6\underline{\mathbf{g}}_{\breve{a}\breve{b}} = \underline{\gamma}_{\breve{a}}\underline{\gamma}_{\breve{b}} + \underline{\gamma}_{\breve{b}}\underline{\gamma}_{\breve{a}}$ with a 6-d Hodge star operator $\underline{\ast}$. These forms obey the conditions

$$J \wedge \underline{\Theta} = 0, \frac{i}{8}\underline{\Theta} \wedge \overline{\underline{\Theta}} = \frac{1}{3!} J \wedge J \wedge J = \underline{\ast}1, \underline{\ast}J = \frac{1}{2} J \wedge J, \underline{\ast}\Theta_{\pm} = \pm\Theta_{\mp},$$

where $\overline{\underline{\Theta}}$ means complex conjugation of $\underline{\Theta}$.

Working with $\tilde{\mathbf{J}}$ instead of J, we can define a similar 3-form $\widetilde{\Theta}$ for an almost-Kähler model $(\widetilde{\theta}, \tilde{\mathbf{J}}, {}_{\theta}\widetilde{\mathbf{D}})$ and construct the Hodge star operator $\widetilde{\ast}$ corresponding to ${}^6\widetilde{\mathbf{g}}$. The relation between 6-d $\widetilde{\ast}$ and 7-d ${}^7\widetilde{\ast}$ Hodge stars for an ansatz of type (26) is

$${}^7\widetilde{\ast}(\ {}^6_p\omega) = e^{B(y^{\breve{a}})}\ \widetilde{\ast}(\ {}^6_p\omega) \wedge \mathbf{e}^4 \text{ and } {}^7\widetilde{\ast}(\mathbf{e}^4 \wedge\ {}^6_p\omega) = e^{-B(y^{\breve{a}})}\ \widetilde{\ast}(\ {}^6_p\omega),$$

where ${}^6_p\omega$ is a p–form with legs only in the directions on ${}^6\mathbf{X}$. The two exterior derivatives ${}^7d$ and $\widetilde{d}$ are related via

$${}^7d(\ {}^6_p\omega) = \widetilde{d}(\ {}^6_p\omega) + dy^4 \wedge \frac{\partial}{\partial y^4}(\ {}^6_p\omega).$$

Applying these formulas, we decompose the 10-d 3-form $\widehat{\mathbf{H}}$ into three N-adapted parts,

$$\widehat{\mathbf{H}} = \ell vol[\ {}^3\mathbf{g}] + \ {}^6\widehat{\mathbf{H}} + dy^4 \wedge \widehat{\mathbf{H}}_4,$$

where

$$vol[\ {}^3\mathbf{g}] = \frac{1}{3}\epsilon_{\check{i}\check{j}\check{k}}\sqrt{|{}^3\mathbf{g}_{\check{i}\check{j}}|}\mathbf{e}^{\check{i}} \wedge \mathbf{e}^{\check{j}} \wedge \mathbf{e}^{\check{k}},\ {}^6\widehat{\mathbf{H}} = \frac{1}{3!}\widehat{\mathbf{H}}_{\breve{a}\breve{b}\breve{c}}\widetilde{\mathbf{e}}^{\breve{a}} \wedge \widetilde{\mathbf{e}}^{\breve{b}} \wedge \widetilde{\mathbf{e}}^{\breve{c}},\ \widehat{\mathbf{H}}_4 = \frac{1}{2!}\widehat{\mathbf{H}}_{4\breve{b}\breve{c}}\widetilde{\mathbf{e}}^{\breve{b}} \wedge \widetilde{\mathbf{e}}^{\breve{c}}.$$

The operators $(\tilde{\mathbf{J}},\widetilde{\Theta})$ allow us to generalize, in almost-Kähler form, the original constructions for Kähler internal spaces provided in [2, 1, 58, 59, 60] for the G_2 structure. In our approach, such an N-adapted configuration is adapted by the data for (27), $(\varpi, \mathcal{W})$ which can be related to the $SU(3)$ almost-Kähler structure $(\tilde{\mathbf{J}},\widetilde{\Theta})$ by expressions

$$\begin{aligned} \varpi &= e^{B(y^{\breve{a}})}\mathbf{e}^4 \wedge \tilde{\mathbf{J}} + \widetilde{\Theta}_-, \\ \mathcal{W} &= e^{B(y^{\breve{a}})}\mathbf{e}^4 \wedge \widetilde{\Theta}_+ + \frac{1}{2}\tilde{\mathbf{J}} \wedge \tilde{\mathbf{J}}. \end{aligned}$$

For any structure group $SU(3)$ and its Lie algebra $\mathfrak{su}(6) = \mathfrak{su}(3) \oplus \mathfrak{su}(3)^{\perp}$, there is a canonical torsion ${}^0\mathbf{T}^{\breve{a}'}_{\ \breve{b}'\breve{c}'} \in \wedge^1 \otimes \mathfrak{su}(3)^{\perp}$, where primed indices refer to an orthonormal basis which can be related to any coordinate and/or N–adapted basis. For instance we write ${}^0\tilde{\mathbf{T}}^{\breve{a}'}_{\ \breve{b}'\breve{c}'}$ if such a basis is for an almost-Kähler structure.

$$\begin{array}{rlllll} {}^0\mathbf{T}^{\breve{a}'}_{\ \breve{b}'\breve{c}'} &= & (3 \oplus \overline{3})\otimes & (1 \oplus 3 \oplus \overline{3}) & & \\ &= & (1 \oplus 1)\oplus & (8 \oplus 8)\oplus & (\overline{6} \oplus 6)\oplus & 2(3 \oplus \overline{3}) \\ & & \mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 & \mathcal{T}_4, \mathcal{T}_5 \end{array}$$

This classification can be N–adapted if we use derivatives of the structure forms

$$\begin{aligned} \widetilde{d}\tilde{\mathbf{J}} &= -\frac{3}{2} Im(\mathcal{T}_1\overline{\widetilde{\Theta}}) + \mathcal{T}_4 \wedge \tilde{\mathbf{J}} + \mathcal{T}_3, \\ \widetilde{d}\widetilde{\Theta} &= \mathcal{T}_1\tilde{\mathbf{J}} \wedge \tilde{\mathbf{J}} + \mathcal{T}_2 \wedge \tilde{\mathbf{J}} + \overline{\mathcal{T}}_5 \wedge \widetilde{\Theta}, \end{aligned}$$

where $Im(\mathcal{T}_1\overline{\widetilde{\Theta}})$ should be treated as the "vertical" part for almost-Kähler structures (when all values are real) and as the imaginary part for complex and Kähler structures $(\tilde{\mathbf{J}}, \widetilde{\Theta}) \to (J, \underline{\Theta})$.

4.4 Nonholonomic instanton d–connections nearly almost Kähler manifolds

The instanton type connections constructed in [2, 1] can be modelled for almost-Kähler internal spaces if we work in N–adapted frames for respective nonholonomically deformed connections. For such configurations, we set

$$\mathcal{T}_2 = \mathcal{T}_3 = \mathcal{T}_4 = \mathcal{T}_5 = 0 \text{ and } \mathcal{T}_1 = \ {}^{+}\mathcal{T}_1 + \tilde{\mathbf{J}}\ {}^{-}\mathcal{T}_1,$$

where the last splitting is defined similarly to (51). All further calculations with $(J, \underline{\Theta})$ in [2, 1, 58, 59, 60] are similar to those for $(\widetilde{\mathbf{J}}, \widetilde{\Theta})$ if we work in N–adapted frames on ${}^{6}\mathbf{X}$ and corresponding model $(\widetilde{\theta}, \widetilde{\mathbf{J}}, {}_{\theta}\widetilde{\mathbf{D}})$. Hereafter, we shall omit detailed proofs for almost-Kähler structures and send readers to analogous constructions in the aforementioned references.

Considering an ansatz (26) with possible embedding into a 10-d one of type (22), with $A = B = 0$ for simpicity, we can construct solutions on ${}^{6}\mathbf{X}$ of the first two nonholonomic BPS equations in (20) and (27). Such almost-Kähler configurations are determined by the system

$$\widehat{\mathbf{H}} = \ell vol[\ {}^{3}\mathbf{g}] - \frac{1}{2}\partial_4\phi + \left(\frac{3}{2}\ {}^{-}\mathcal{T}_1 + \frac{7}{8}\ell\right) + dy^4 \wedge (2\ {}^{-}\mathcal{T}_1 + \ell)\widetilde{\mathbf{J}} \text{ for } \widehat{\phi} = \phi(y^4) \quad (52)$$

and $(\tilde{\mathbf{J}}, \widetilde{\Theta})$ subjected to respective flow and structure equations:

$$\begin{aligned} \partial_4 \widetilde{\mathbf{J}} &= (\ {}^{+}\mathcal{T}_1 + \partial_4\phi)\widetilde{\mathbf{J}}, \\ \partial_4\ {}^{-}\widetilde{\Theta} &= -(3\ {}^{-}\mathcal{T}_1 + \frac{15}{8}\ell)\ {}^{+}\widetilde{\Theta} + \frac{3}{2}(\ {}^{+}\mathcal{T}_1 + \partial_4\phi)\ {}^{-}\widetilde{\Theta}, \\ \partial_4\ {}^{+}\widetilde{\Theta} &= \frac{3}{2}(\ {}^{+}\mathcal{T}_1 + \partial_4\phi)\ {}^{+}\widetilde{\Theta} + \tilde{\alpha}(y^4)\ \ {}^{-}\widetilde{\Theta}, \text{ for arbitrary function } \tilde{\alpha}(y^4); \end{aligned} \quad (53)$$

$$\begin{aligned} \text{and } \widetilde{d}\widetilde{\mathbf{J}} &= -\frac{3}{2}\ {}^{-}\mathcal{T}_1\ {}^{+}\widetilde{\Theta} + \frac{3}{2}\ {}^{+}\mathcal{T}_1\ {}^{-}\widetilde{\Theta}, \\ \widetilde{d}\widetilde{\Theta} &= \mathcal{T}_1 \widetilde{\mathbf{J}} \wedge \widetilde{\mathbf{J}}. \end{aligned} \quad (54)$$

At the zeroth order in α' with the Bianchi identity $\widehat{\mathbf{d}}\widehat{\mathbf{H}} = 0$, see (19), we can choose $\widehat{\mathbf{F}} = 0$ in order to solve the third equation in (20). The time–like component of the equations of motion can be solved if $\ell = 0$ as in the pure Kähler case [1]. We obtain a special case of solutions (see [58] for original Kähler ones) when

$$\begin{array}{lllll} 1. & \phi = const., & {}^{+}\mathcal{T}_1 \text{ is a free function }, & {}^{-}\mathcal{T}_1 = 0, & \tilde{\alpha}(y^4) \text{ is a free function }; \\ 2. & \phi = \frac{2}{3}\log(a_0 y^4 + b_0), & {}^{+}\mathcal{T}_1 = 0, & {}^{-}\mathcal{T}_1 = 0, & \tilde{\alpha} = 0, \end{array} \quad (55)$$

for integration constants a_0 and b_0 corresponding to N–adapted frames. Respectively, cases 1 and 2 correspond to a nearly almost-Kähler geometry, with nonholonomically induced torsion (by off–diagonal N-terms) and vanishing NS 3-form flux, and a nonholonomic generalized Calabi-Yau with flux.

5 The YM Sector and Nonholonomic Heterotic Supergravity

In this section, we construct a nontrivial gauge d–field $\widehat{\mathbf{F}}$ which arises at the first order α', when the YM sector can not be ignored. The approach elaborates a noholonomic and almost-Kähler version of YM instantons studied in [2, 58, 1]. The equations of motion of heterotic supergravity are then re–written in canonical nonholonomic variables, which allows us to decouple and find general integrals of such systems using methods applied for nonholonomic EYMH and Einstein–Dirac fields, see [30, 32, 36, 37, 38, 47, 48].

5.1 N–adapted YM and instanton configurations

The nonholonomic instanton equations can be formulated on ${}^{7}\mathbf{X} = \mathbb{R}\times {}^{6}\mathbf{X}$ with generalized 'h–cone' d-metric

$$\begin{aligned} {}^{7}_{c}\mathbf{g} &= (\mathbf{e}^{4})^{2} + [\ {}_{|}h(y^{4})]^{2}\ {}^{6}\mathbf{g}(y^{\check{c}}) = (\mathbf{e}^{4})^{2} + [\ {}_{|}h(y^{4})]^{2}\ \ {}^{6}\widetilde{\mathbf{g}}_{\check{a}\check{b}}(y^{\check{c}}), \\ \mathbf{e}^{4} &= dy^{4} + w_{i}(x^{k}, y^{a}), \end{aligned} \tag{56}$$

where ${}^{6}\mathbf{g}$ (28) can be considered for exact solutions determined in 10-d gravity and ${}^{6}\widetilde{\mathbf{g}}_{\check{a}\check{b}} = [\widetilde{g}_{\acute{i}\acute{j}}, \widetilde{g}_{\grave{a}\grave{b}}, \tilde{N}^{\grave{a}}_{i}]$ (31) is for elaborating respective almost Kähler models.[8] We denote by ${}_{|}\mathbf{e}^{\widetilde{a}\prime} = \{\mathbf{e}^{4\prime},\ {}_{|}h \cdot\ {}_{|}\widetilde{\mathbf{e}}^{\check{a}\prime}\} \in T^{*}(\ {}^{7}\mathbf{X})$ an orthonormal N–adapted basis on $\widetilde{a}', \widetilde{b}', ... = 4,5,6,7,8,9,10,$ with ${}^{|}\widetilde{\mathbf{e}}^{\check{a}\prime} = e^{\check{a}\prime}_{\check{a}}\widetilde{\mathbf{e}}^{\check{a}}$, when ${}^{6}\widetilde{\mathbf{g}}_{\check{a}\check{b}} = \delta_{\check{a}\prime\check{b}\prime} e^{\check{a}\prime}_{\check{a}} e^{\check{b}\prime}_{\check{b}}$.

We can relate the N–adapted configuration to the orthonormal frame $\mathbf{e}^{\check{a}\prime}$ by introducing certain Kähler operators in standard form instead of $(\tilde{\mathbf{J}}, \widetilde{\Theta})$,

$$\begin{aligned} {}_{|}\tilde{\mathbf{J}} &:= \ {}_{|}\widetilde{\mathbf{e}}^{5} \wedge\ {}_{|}\widetilde{\mathbf{e}}^{6} +\ {}_{|}\widetilde{\mathbf{e}}^{7} \wedge\ {}_{|}\widetilde{\mathbf{e}}^{8} +\ {}_{|}\widetilde{\mathbf{e}}^{9} \wedge\ {}_{|}\widetilde{\mathbf{e}}^{10}, \\ {}_{|}\widetilde{\Theta} &:= (\ {}_{|}\widetilde{\mathbf{e}}^{5} + i\ {}_{|}\widetilde{\mathbf{e}}^{6}) \wedge (\ {}_{|}\widetilde{\mathbf{e}}^{7} + i\ {}_{|}\widetilde{\mathbf{e}}^{8}) \wedge (\ {}_{|}\widetilde{\mathbf{e}}^{9} +\ {}_{|}\widetilde{\mathbf{e}}^{10}), \end{aligned}$$

where $i^{2} = -1$ is used for $SU(3)$. Using properties of such orthonormalized N-adapted bases, we can verify that standard conditions for the nearly Kähler internal spaces are satisfied [2, 58, 1], but also mimic almost-Kähler manifolds with

$$d(\ {}_{|}^{+}\widetilde{\Theta}) = 2\ \ {}_{|}\tilde{\mathbf{J}} \wedge\ {}_{|}\tilde{\mathbf{J}} \text{ and } d\ {}_{|}\tilde{\mathbf{J}} = 3(\ {}_{|}^{-}\widetilde{\Theta}).$$

In result, we can consider the same reduction of the instanton equations (third formula in (20)) as for holonomic Kähler structures with nontrivial torsion structure encoded in N–adapted bases for differential forms on nonholonomic internal manifolds ${}^{7}\mathbf{X}$. Using two types of warping variables,

$$dy^{4} = e^{f(\tau)}d\tau, \text{ for } e^{f(\tau)} =\ {}_{|}h(y^{4}(\tau)) \tag{57}$$

and two equivalent d–metrics,

$${}^{7}_{c}\mathbf{g} = e^{2f}\ {}^{7}_{z}\mathbf{g} \text{ with } {}^{7}_{z}\mathbf{g} = d\tau^{2} +\ {}^{6}\widetilde{\mathbf{g}}_{\check{a}\check{b}},$$

we obtain nonholonomic instanton equations

$$*_{z}\ \widetilde{\mathbf{F}} = -(*_{z}\mathbf{Q}_{z}) \wedge \widetilde{\mathbf{F}}, \tag{58}$$

where $\mathbf{Q}_{z} = d\tau \wedge\ {}_{|}^{+}\widetilde{\Theta} + \frac{1}{2}\ {}_{|}\tilde{\mathbf{J}} \wedge\ {}_{|}\tilde{\mathbf{J}}$ and $*_{z}$is the Hodge star with respect to the cylinder metric ${}^{7}_{z}\mathbf{g}$. The almost-Kähler structure of ${}^{7}\mathbf{X}$ is encoded into boldface operators $\mathbf{Q}_{z}$, ${}_{|}\tilde{\mathbf{J}}$ and canonical tilde like for ${}_{|}^{+}\widetilde{\Theta}$. This does not allow us to solve such equations with an ansatz for the canonical connection on ${}^{6}\mathbf{X}$ determined by the LC–connection as in [2, 1] but imposes the necessity to involve the normal (almost symplectic) d–connection $\widetilde{\mathbf{D}}_{\check{a}} = (\widetilde{D}_{\acute{k}}, \widetilde{D}_{\grave{b}}) = \{\widetilde{\omega}^{\check{b}\prime}_{\ \check{a}\check{c}\prime}\}$, (42). Let us consider

$${}_{A}\widetilde{\mathbf{D}} =\ {}^{can}\widetilde{\mathbf{D}} + \psi(\tau)\ {}_{|}\mathbf{e}^{\widetilde{a}\prime} I_{\widetilde{a}\prime},$$

where the canonical d–connection on ${}^{6}\mathbf{X}$ enabled with almost-Kähler structure is

$${}^{can}\widetilde{\mathbf{D}} = \{\ {}^{can}\widetilde{\omega}^{\check{b}\prime}_{\ \check{a}\check{c}\prime} := \widetilde{\omega}^{\check{b}\prime}_{\ \check{a}\check{c}\prime} + \frac{1}{2}(\ {}_{|}^{+}\widetilde{\Theta})^{\check{b}\prime}_{\ \check{c}\prime\check{a}\prime} e^{\check{a}\prime}_{\check{a}}\}.$$

[8] The warping factor $h(y^{4})$ can be considered in a more generalized form $h(x^{i}, y^{3}, y^{4})$ because the AFDM also allows us to generate these classes of solutions. For simplicity, we shall consider factorizations and frame transforms when the warping factor depends only on the coordinate y^{4}. This allows us to reproduce, in explicit form, the results for Kähler internal spaces if the 6-d metric structures do not depend on y^{4}.

In these formulas, the matrices $I_{\widetilde{a}'} = (I_{\widetilde{i}'}, I_{\check{a}'})$ split into a basis/generators $I_{\widetilde{i}'} \subset \mathfrak{so}(3)$ and generators $I_{\check{a}'}$ for the orthogonal components of $\mathfrak{su}(3)$ in $\mathfrak{g} \subset \mathfrak{so}(7)$ satisfy the Lie algebra commutator

$$[I_{\check{a}'}, I_{\check{b}'}] = f^{\widetilde{i}'}_{\check{a}'\check{b}'} I_{\widetilde{i}'} + f^{\check{c}'}_{\check{a}'\check{b}'} I_{\check{c}'},$$

with respective structure constants $f^{\widetilde{i}'}_{\check{a}'\check{b}'}$ and $f^{\check{c}'}_{\check{a}'\check{b}'}$ (see formulas (3.9) in [1] for explicit parametrizations).

We can define and compute the curvature d–form

$$\begin{aligned} {}_A\widetilde{\mathbf{F}} &= \frac{1}{2}[\ {}_A\widetilde{\mathbf{D}},\ {}_A\widetilde{\mathbf{D}}] := \mathcal{F}(\psi) \\ &= {}^{can}\widetilde{R} + \frac{1}{2}\psi^2 f^{\widetilde{i}'}_{\check{a}'\check{b}'}\ I_{\widetilde{i}'}\ {}_{\shortmid}\mathbf{e}^{\check{a}'} \wedge\ {}_{\shortmid}\mathbf{e}^{\check{b}'} + \frac{\partial\psi}{\partial\tau} d\tau \wedge I_{\check{c}'}\ {}_{\shortmid}\mathbf{e}^{\check{c}'} + \frac{1}{2}(\psi - \psi^2) I_{\check{b}'}(\ {}_{\shortmid}^{+}\widetilde{\Theta})^{\check{b}'}{}_{\check{c}'\check{a}'}\ {}_{\shortmid}\mathbf{e}^{\check{c}'} \wedge\ {}_{\shortmid}\mathbf{e}^{\check{a}'}, \end{aligned}$$

with parametric dependence on τ (57) via $\psi(\tau)$. Such an ${}_A\widetilde{\mathbf{F}}$ is a solution of the nonholonomic instanton equations (58) for any solution of the 'kink equation'

$$\frac{\partial\psi}{\partial\tau} = 2\psi(\psi - 1).$$

For the aforementioned types of d–connections, and for the LC-connection, we can consider two fixed points, $\psi = 0$ and $\psi = 1$ and a non-constant solution

$$\psi(\tau) = \frac{1}{2}\left(1 - \tanh|\tau - \tau_0|\right),$$

where the integration constant τ_0 fixes the position of the instanton in the τ direction but such an instanton also encodes an almost-Kähler structure.

In heterotic supergravity, we can consider two classes of nonholonomic instanton configurations. The first one is for the gauge-like curvature, ${}_A\widetilde{\mathbf{F}} = \mathcal{F}(\ {}^1\psi)$ and $\widetilde{\mathbf{R}} = \mathcal{R}(\ {}^2\psi)$, [in the second case, we also solve the condition $\widetilde{\mathbf{R}} \cdot \epsilon = 0$] where the values ${}^1\psi$ and ${}^2\psi$ will be defined below.

5.2 Static and/or dynamic $SU(3)$ nonholonomic structures on almost Kähler configurations

The transforms ${}^6\widetilde{\mathbf{g}}_{\check{a}\check{b}} = [\ {}_{\shortmid}h]^2\ \ {}^6\widetilde{\mathbf{g}}_{\check{a}\check{b}}$ in d–metric (56) impose certain relations on the two pairs $(\ {}_{\shortmid}\tilde{\mathbf{J}},\ {}_{\shortmid}\widetilde{\Theta})$ and $(\tilde{\mathbf{J}},\widetilde{\Theta})$, where the last couple is subjected to the respective flow and structure equations, (53) and (54), and define the 3–form $\widehat{\mathbf{H}}$ (52). We note that such relations are a mixing between real and imaginary parts and nonholonomically constrained in order to adapt the Lie algebra symmetries to the almost-Kähler structure. Like in [1], it is considered a y^4 depending mixing angle ${}_{\shortmid}\beta(y^4) \in [0, 2\pi)$ when

$$\begin{aligned} \tilde{\mathbf{J}} &= [\ {}_{\shortmid}h]^2\ {}_{\shortmid}\tilde{\mathbf{J}}, \\ {}^{+}\widetilde{\Theta} &= [\ {}_{\shortmid}h]^3(\ {}_{\shortmid}^{+}\widetilde{\Theta}\cos\ {}_{\shortmid}\beta +\ {}_{\shortmid}^{-}\widetilde{\Theta}\sin\ {}_{\shortmid}\beta),\ \ {}^{-}\widetilde{\Theta} = [\ {}_{\shortmid}h]^3(-\ {}_{\shortmid}^{+}\widetilde{\Theta}\sin\ {}_{\shortmid}\beta +\ {}_{\shortmid}^{-}\widetilde{\Theta}\cos\ {}_{\shortmid}\beta). \end{aligned}$$

Introducing such values into the relations for $(\tilde{\mathbf{J}},\ {}^{+}\widetilde{\Theta},\ {}^{-}\widetilde{\Theta})$, we obtain (compare to (55))

$$\begin{aligned} \widehat{\mathbf{H}} &= \ell vol[\ {}^3\mathbf{g}] +\ {}_{\shortmid}h dy^4 \wedge (\ell\ {}_{\shortmid}h - 4\sin\ {}_{\shortmid}\beta)\ {}_{\shortmid}\tilde{\mathbf{J}} \\ &\quad +[\ {}_{\shortmid}h]^2\left[-\frac{1}{2}\ {}_{\shortmid}h(\partial_4\phi)\cos\ {}_{\shortmid}\beta + 3\sin^2\ {}_{\shortmid}\beta - \frac{7}{8}\ell\ {}_{\shortmid}h\sin\ {}_{\shortmid}\beta\right]\ {}_{\shortmid}^{+}\widetilde{\Theta} \\ &\quad +[\ {}_{\shortmid}h]^2\left[-\frac{1}{2}\ {}_{\shortmid}h(\partial_4\phi)\sin\ {}_{\shortmid}\beta - 3\sin\ {}_{\shortmid}\beta\cos\ {}_{\shortmid}\beta + \frac{7}{8}\ell\ {}_{\shortmid}h\cos\ {}_{\shortmid}\beta\right]\ {}_{\shortmid}^{-}\widetilde{\Theta}. \end{aligned}$$

The above formula involves the conditions

$$^{+}\mathcal{T}_1 = 2(\,{}_{\shortmid}h)^{-1}\cos\,{}_{\shortmid}\beta \text{ and } ^{-}\mathcal{T}_1 = -2(\,{}_{\shortmid}h)^{-1}\sin\,{}_{\shortmid}\beta$$

which allows us to fix

$$\tilde{\alpha}(y^4) = 3\ ^{-}\mathcal{T}_1 + \frac{15}{8}\ell.$$

There are additional conditons on the scalar functions ${}_{\shortmid}h,\ {}_{\shortmid}\beta, \ell$ and ϕ which must be imposed on coupled nonholonomic instanton solutions $\ ^1\psi$ and $\ ^2\psi$ satisfying the Bianchi conditions, the nonholonomic BPS equations and the time–like components of the equations of motion. We omit such an analysis because it is similar to that of the pure Kähler configurations, see sections 4 and 5 in [1]. The priority of nonholonomic almost-Kähler variables is so that we can work with respect to N-adapted frames in a form which is very similar to that for complex and symplectic structures.

5.3 Equations of motion of heterotic supergravity in nonholonomic variables

In order to apply the AFDM, we have to rewrite the motion equations (generalized Einstein equations) in nonholonomic variables. The formal procedure is to take such equations written for the LC–connection with respect to coordinate frames and re–write them for the same metric structure, but for corresponding geometric objects with "hats " and "waves" and with respect to N–adapted frames on corresponding shells. In this way, including terms of order α', the N–adapted equations of motion of heterotic nonholonomic supergravity considered in [1] can be written in such a form:

$$\widehat{\mathbf{R}}_{\mu_s\nu_s} + 2(\ ^s\widehat{\mathbf{D}}\widehat{\mathbf{d}}\widehat{\phi})_{\mu_s\nu_s} - \frac{1}{4}\widehat{\mathbf{H}}_{\alpha_s\beta_s\mu_s}\widehat{\mathbf{H}}_{\nu_s}^{\ \ \alpha_s\beta_s} + \frac{\alpha'}{4}\left[\widetilde{\mathbf{R}}_{\mu_s\alpha_s\beta_s\gamma_s}\widetilde{\mathbf{R}}_{\nu_s}^{\ \ \alpha_s\beta_s\gamma_s} - tr\left(\widehat{\mathbf{F}}_{\mu_s\alpha_s}\widehat{\mathbf{F}}_{\nu_s}^{\ \ \alpha_s}\right)\right] = 0, \quad (59)$$

$$^s\widehat{R} + 4\widehat{\square}\widehat{\phi} - 4|\widehat{\mathbf{d}}\widehat{\phi}|^2 - \frac{1}{2}|\widehat{\mathbf{H}}|^2 + \frac{\alpha'}{4}tr\left[|\widetilde{\mathbf{R}}|^2 - |\widehat{\mathbf{F}}|\right] = 0, \quad (60)$$

$$e^{2\widehat{\phi}}\widehat{\mathbf{d}}\widehat{*}(e^{-2\widehat{\phi}}\widehat{\mathbf{F}}) + \widehat{\mathbf{A}}\wedge\widehat{*}\widehat{\mathbf{F}} - \widehat{*}\widehat{\mathbf{F}}\wedge\widehat{\mathbf{A}} + \widehat{*}\widehat{\mathbf{H}}\wedge\widehat{\mathbf{F}} = 0, \quad (61)$$

$$\widehat{\mathbf{d}}\widehat{*}(e^{-2\widehat{\phi}}\widehat{\mathbf{H}}) = 0, \quad (62)$$

where the Hodge operator $\widehat{*}$, $\ ^s\widehat{\mathbf{D}} = \{\widehat{\mathbf{D}}_{\mu_s}\}$ (12), the canonical nonholonomic d'Alambert wave operator $\widehat{\square} := \widehat{\mathbf{g}}^{\mu_s\nu_s}\widehat{\mathbf{D}}_{\mu_s}\widehat{\mathbf{D}}_{\nu_s}$, $\widehat{\mathbf{R}}_{\mu_s\nu_s}$ (17), $^s\widehat{R}$ (18) are all determined by a d–metric $\widehat{\mathbf{g}}$ (10). The curvature d–tensor $\widetilde{\mathbf{R}}_{\mu_s\alpha_s\beta_s\gamma_s}$ is taken for an almost-Kähler structure $\widetilde{\theta}$ (40) defined by corresponding nonholonomic distributions which are stated up to frame transforms by the N–connection structure and components of d–metric on shells $s = 1,2,3$ as we described above. The gauge field $\widehat{\mathbf{A}}$ corresponds to the N–adapted operator

$$^s_A\widehat{\mathbf{D}} = \ ^s\widehat{\mathbf{D}} + \ ^1\psi(y^4)[\mathbf{e}^{a_1}I_{a_1} + \mathbf{e}^{a_2}I_{a_2} + \mathbf{e}^{a_3}I_{a_3}] = \ ^s\widehat{\mathbf{D}} + \ ^1\psi(y^4)I_{\check{c}'}\ {}_{\shortmid}\mathbf{e}^{\check{c}'} = \widehat{\mathbf{d}} + \widehat{\mathbf{A}}$$

and curvature $\widehat{\mathbf{F}} = \mathcal{F}(\ ^1\psi)$ via a map constructed above (for $^s\widehat{\mathbf{D}}_{|\widehat{\mathcal{T}}=0} \to \ ^s\nabla$, see details in [1]). For instance, the LC–configurations of (59) are determined by equations

$$R_{\mu\nu} + 2(\nabla d\phi)_{\mu\nu} - \frac{1}{4}H_{\alpha\beta\mu}H_{\nu}^{\ \ \alpha\beta} + \frac{\alpha'}{4}\left[\widetilde{R}_{\mu\alpha\beta\gamma}\widetilde{R}_{\nu}^{\ \ \alpha\beta\gamma} - tr\left(\widehat{F}_{\mu\alpha}\widehat{F}_{\nu}^{\ \ \alpha}\right)\right] = 0 \quad (63)$$

with standard 10-d indices $\alpha, \beta, ... = 0,1,2,...9$ and geometric values determined by ∇. Unfortunately, the system of nonlinear PDEs (63) can not be decoupled and integrated in any general form if we do not consider shell N–adapted frames/coordinates and generalized connections which can be nonholonomically constrained to LC-configurations.

The equations (61) and (62) can be solved for arbitrary almost-Kähler internal spaces by introducing corresponding classes of N-adapted variables as we proved in previous sections. Such solutions can be classified as for pure Kähler spaces, for simplicity, considering a special case with $\ ^{-}\mathcal{T}_1 = 0$ and $\ell = 0$ and in terms of 'kink' solutons with

$$e^{2f} = e^{2(\tau-\tau_0)} + \frac{\alpha'}{4}[(\ ^1\psi)^2 - (\ ^2\psi)^2], \tau_0 = const.$$

The NS 3–form flux is given by a simple formula

$$\widehat{\mathbf{H}}(\tau, y^{\check{c}}) = -\frac{1}{2} = [\ _{\shortmid}h]^3(\partial_4\phi)(\ _{\shortmid}^{+}\widetilde{\Theta}) = \frac{\alpha'}{4}[(\ ^1\psi)^2(2\ ^1\psi - 3) - (\ ^2\psi)^2(2\ ^2\psi - 3)](\tau)[\ _{\shortmid}^{+}\widetilde{\Theta}(y^{\check{c}})]. \quad (64)$$

Here we reproduce the classification of 8 cases with fixed and/or kink configurations for almost-Kähler configurations,

Case	$^1\psi,\ ^2\psi$;	$e^{2f} = e^{2(\tau-\tau_0)} + \frac{\alpha'}{4}[(\ ^1\psi)^2 - (\ ^2\psi)^2], \phi(\tau) = \phi_0 + 2(f - \tau)$
1.	$^1\psi =\ ^2\psi$;	$f = \tau - \tau_0, \phi = \phi_0 - 2\tau$
2.	$^1\psi =\ ^2\psi = 0$;	$f = f_0 := \frac{1}{2}\log(\frac{\alpha'}{4}), \phi = \phi_0 + 2(f_0 - \tau)$
3.	$^1\psi = 1,\ ^2\psi = 0$;	$e^{2f} = e^{2(\tau-\tau_0)} - \frac{\alpha'}{4}, e^{\phi-\phi_0} = e^{-2\tau_0} - \frac{\alpha'}{4}e^{-2\tau}$
4.	$^1\psi = 0,\ ^2\psi =$ kink ;	$e^{2f} = e^{2(\tau-\tau_0)} + \frac{\alpha'}{16}[1 - \tanh(\tau - \tau_1)]^2, \phi = \phi_0 + 2(f - \tau)$
5.	$^1\psi =$ kink , $^2\psi = 0$;	$e^{2f} = e^{2(\tau-\tau_0)} - \frac{\alpha'}{16}[1 - \tanh(\tau - \tau_1)]^2, \phi = \phi_0 + 2(f - \tau)$
6.	$^1\psi = 1,\ ^2\psi =$ kink ;	$e^{2f} = e^{2(\tau-\tau_0)} + \frac{\alpha'}{16}[\tanh(\tau - \tau_1) + 1][\tanh(\tau - \tau_1) - 3]$
7.	$^1\psi =$ kink , $^2\psi = 1$;	$e^{2f} = e^{2(\tau-\tau_0)} - \frac{\alpha'}{16}[\tanh(\tau - \tau_1) + 1][\tanh(\tau - \tau_1) - 3]$
8.	$^1\psi =$ kink , $^2\psi =$ kink with $\tau_1 \neq \tau_2$;	$e^{2f} = e^{2(\tau-\tau_0)} + \frac{\alpha'}{16}[\tanh^2(\tau - \tau_1) - 2\tanh(\tau - \tau_1) - \tanh^2(\tau - \tau_2) + 2\tanh(\tau - \tau_2)]$

Such a classification can be used for parametrizing certain effective sources of Einstein–Yang-Mills-Higgs type and preserved for constructing generic off-diagonal solutions following the AFDM [30, 31, 34, 29, 36, 37, 23, 24]. We emphasize that classes 1-8 distinguish different almost-Kähler structures encoding corresponding assumptions, that for identification of almost complex structure I with the complex unity i in the typical fiber of tangent space use the same classification as for symplectic configurations introduced in [1, 2]. A unified classification for internal (almost) Kähler spaces is possible with respect to corresponding N–adapted frames generated by a conventional Lagrange function as we considered in formulas (30) and (40).

Finally, we note that we shall construct and analyze various classes of generic off-diagonal exact solutions of the equations (59) and (60) in our associated work [3]. The main goal of that work is to prove that it is possible to reproduce certain types of off–diagonal deformations of the Kerr metric in heterotic supergravity in the 4-d sector, if the internal 6-d space is endowed with richer (than in Kähler geometry) structures. In section 4, we proved that it is always possible to introduce such nonholonomic variables when the internal space geometric data is parameterized via geometric objects of an effective almost-Kähler geometry. This allows us to self-consistently solve the equations (61) and (62) following the same procedure and classification as in [1, 2] when the domain walls were endowed with trivial pseudo-Euclidean structure warped nearly Kähler internal spaces. The off–diagonal deformation techniques defined by the AFDM allow us to generalize the constructions for nontrivial exact and parametric solutions in 4-d, 6-d and 10-d MGTs and string gravity.

6 Conclusions

In this work, we have studied how to generalize certain classes of 'prime' solutions in heterotic string gravity constructed in [1, 2] for arbitrary 'target' ten dimensional, 10-d, metrics. The structure and classification of prime configurations [with pseudo-Euclidean (1+3)–dimensional domain walls and 6-d warped nearly Kähler manifolds in the presence of gravitational and gauge instantons] can be preserved for nontrivial curved 4-d spacetime configurations. For instance, we can generate black hole/ellipsoid configurations if certain types of nonholonomic variables with conventional 2+2+....=10 splitting are defined and respective almost-Kähler internal configurations are associated for corresponding 2+2+2=3+3 double fibrations. The diadic "shell by shell" nonholonomic decomposition of 4-d, 6-d and 10-d pseudo-Riemannian manifolds allows us to integrate the motion equations (59)–(62) in very general forms using the AFDM, see further developments of this paper in [3]. The 3+3 decomposition is used for constructing nonholonomic deformed instanton configurations which are necessary for solving the Yang-Mills sector and the generalized Bianchi identity at order α', when certain generalized classes of solutions may contain internal configurations depending, in principle, on all 9 space-like coordinates for a 10-d effective gravity theory. The triadic formalism is also important for associating $SU(3)$ structures in certain holonomic limits to well known solutions (with more "simple" domain wall and an internal structures) in heterotic string gravity.

The geometric techniques elaborated in this paper (which is a development for the heterotic string gravity results of a series of studies [29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 9, 10, 40]) allows us to work with arbitrary stationary generic off-diagonal metrics on 10-d spacetimes. Effective MGTs spacetimes can be enabled with generalized connections, depending on all possible 4-d and extra dimension space coordinates. One of main results of the presented work is that the system (61)–(62) admits subclasses of solutions with warping on coordinate y^4 nearly almost-Kähler 6-d internal manifolds in the presence of nonholonomically deformed gravitational and gauge instantons. The almost-Kähler structure is necessary if we want to generate in the 4-d spacetime part, for instance, the Kerr metric with possible (off-) diagonal and nonholonomic deformations to black ellipsoid type configurations characterized by locally anisotropic polarized physical constants, small deformations of horizons, embedding into nontrivial extra dimension vacuum gravitational fields and/or gauge configurations [which are considered in details in [29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 3]]. In this work, we concentrate on 10-d configurations preserving two real supercharges corresponding to $N = 1/2$ supersymmetry from the viewpoint of four non-compact dimensions and various nonholonomic deformations.

Following methods of the geometry of nonholonomic manifolds and almost-Kähler spaces, applied for deformation and/or A-brane quantization and geometric flows of gravity theories [9, 10, 40], we defined two pairs of related $({}_{\shortmid}\tilde{\mathbf{J}}, {}_{\shortmid}\widetilde{\Theta})$ and $(\tilde{\mathbf{J}},\widetilde{\Theta})$ structures depending both on warping and other space coordinates. Such constructions encode possible almost symplectic configurations determined by the 6-d internal sector of heterotic gravity and BPS equations re-written in nonholonomic variables. This involves and mixes certain gauge like, i.e. a NS 3-form for flux, gravitational solitons, and effective scalar fields. For additional constraints, such stationary configurations transform into static $SU(3)$ structures as in [1, 2]. This assumption is crucial for classifying new types of nonolonomically deformed solutions following the same principles, as it was done originally for Kähler internal spaces and standard instanton constructions.

Finally, we note that it remains to be seen how the AFDM generates certain classes of physically important solutions of the motion equations (59)–(60) (like black holes, cosmological metrics etc.) in the 4-d sector, using the nonholonomic solutions of (61)–(62) generated in this work. In the associated paper [3], the motion equations in heterotic string gravity resulting in stationary metrics, in particular, in generic off-diagonal deformations of the Kerr solution to certain ellipsoid like configurations are solved by integrating in very general off-diagonal forms. Further developments in string MGTs with cosmological solutions of type [11, 12, 15, 16, 19, 25, 26, 27, 28] are left for future work.

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